Statistics 522, Problem Set 2

Wellner; 1/23/2008

Reading:

Shorack, PfS,	Chapter 11, Sections 7-8, pages 312 - 318;
Shorack, PfS,	Chapter 13, Sections 1-7, pages 367 - 390.

Due: Wednesday, January 30, 2008.

1. (Symmetry and conditional expectation). Let $X_1, X_2, \ldots$ be i.i.d. random variables with the same distribution as X where $E|X| < \infty$. Let $S_n \equiv X_1 + \cdots + X_n$, and define

$$\mathcal{G}_n \equiv \sigma\left[S_n, S_{n+1}, \ldots\right] = \sigma\left[S_n, X_{n+1}, X_{n+2}, \ldots\right].$$

Show that $E(X_1|S_n) = n^{-1}S_n$ almost surely. [Hint: Note that $\sigma\left[X_{n+1}, X_{n+2}, \ldots\right]$ is independent of $\sigma\left[X_1, S_n\right]$, and use symmetry to show that $E(1_{[S_n \in B]}X_1) = E(1_{[S_n \in B]}X_2) = \cdots = E(1_{[S_n \in B]}X_n)$.]

2. (An application of the WLLN: Inversion of Laplace transforms) Let P be a probability measure on the Borel subsets of $[0, \infty)$, and define its *Laplace transform* by $\varphi(t) = \int_0^\infty e^{-tx} dP(x)$ for $t \in [0, \infty)$. Widder's inversion formula for P from φ is:

$$\lim_{n \to \infty} \sum_{k=0}^{[nz]} \frac{(-1)^k}{k!} n^k \varphi^{(k)}(n) = P([0, z)) \tag{1}$$

Show that (1) holds via the following steps:
(a) Differentiation of the integral k times shows that

$$\varphi^{(k)}(t) = \int_0^\infty (-x)^k e^{-tx} dP(x).$$

(b) Setting $t = n$, letting $z > 0$, multiplying across by $(-1)^k n^k / k!$, and summing on k yields

$$\sum_{k=0}^{[nz]} \frac{(-1)^k}{k!} n^k \varphi^{(k)}(n) = \int_0^\infty \sum_{k=0}^{[nz]} e^{-nx} \frac{(nx)^k}{k!} dP(x) \tag{2}$$

where $e^{-nx}(nx)^k/k! = P(S_n = k)$ and $S_n = Y_1 + \cdots + Y_n$ where $Y_1, Y_2, \ldots$ are i.i.d. Poisson(x).
(c) Use the weak law of large numbers and (2) to show that (1) holds.

3. PfS, Exercise 8.6.1, page 188.

4. PfS, Exercise 8.6.2, page 189.

5. Suppose that $X, Y \in L_1(\Omega, \mathcal{F}, P)$ and that $E(Y|X) = X$ a.s. and $E(X|Y) = Y$ a.s. Prove that $P(X = Y) = 1$. (See e.g. Exercise 9.2, Williams, *Probability with Martingales*, page 231.)