Statistics 522, Midterm Exam

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1. (24 points). **Define** *three* of the following four terms:
   (a) The conditional expectation of a random variable X given a (sub-) sigma-field $\mathcal{D}$.
   (b) Convergence in distribution of a sequence of measures $\{P_n\}$ on a metric space (M, d) to a measure P.
   (c) The class of bounded Lipschitz functions $BL(M)$ on a metric space (M, d).
   (d) The Lévy distance $\lambda(F, G)$ between two distribution functions F and G on $\mathbb{R}$.

2. (24 points). Give careful **statements** of *any two* of the following four theorems or results:
   (a) The law of the iterated logarithm for i.i.d. random variables $X_1, X_2, \ldots$ with $E(X_1) = 0$ and $Var(X_1) = \sigma^2$.
   (b) The Lindeberg-Feller central limit theorem for a triangular array of row-wise independent random variables with $E(X_{n,i}) = 0$ and $Var(X_{n,i}) = \sigma_{n,i}^2 < \infty$, $i = 1, \ldots, n$.
   (c) The interpretation of conditional expectation $E(Y|\mathcal{D})$ for $Y \in L_2(P)$ as an (orthogonal) projection onto $L_2(\Omega, \mathcal{D}, P)$ where $\mathcal{D} \subset \mathcal{A}$.
   (d) Any CLT result with a rate derived from Stein's method.

3. (24 points). Let X and Y be two random variables defined on a probability space $(\Omega, \mathcal{A}, P)$ with $E(Y^2) < \infty$. Define $Var(Y|X) = E\{(Y - E(Y|X))^2|X\}$. Show that

$$Var(Y) = EVar(Y|X) + Var(E(Y|X)).$$

4. (36 points). Suppose that conditionally on Λ the random variable W has a Poisson distribution where $\Lambda \sim G$ on $\mathbb{R}^+$. That is,

$$P(W = k) = EP(W = k|\Lambda) = \int_0^\infty e^{-v}\frac{v^k}{k!}dG(v), \qquad k \in \{0, 1, 2, \ldots\}.$$

   (a) What is the equation characterizing the Poisson(λ) distribution upon which the Stein-Chen approximation method is based?
   (b) Show that for each fixed $\lambda \in \mathbb{R}$ the following bound is valid:

$$d_{TV}(P_W, P_\lambda) \le \min\{1, \lambda^{-1/2}\}E|\Lambda - \lambda|.$$

   Here P_λ is the Poisson probability measure on $\mathbb{Z}^+$ with parameter λ: for any $A \subset \mathbb{Z}^+$,

$$P_\lambda(A) = \sum_{z \in A} e^{-\lambda}\frac{\lambda^z}{z!}.$$

Hint: Condition on Λ and use the fact that $\|g\|_\infty \leq \min\{1, \lambda^{-1/2}\}$ for any $g = g_{\lambda,A}$ with $A \subset \mathbb{Z}^+$ solving the Stein-Chen equation $\lambda g(z+1) - zg(z) = 1_A(z) - P_\lambda(A)$ for $z \in \mathbb{Z}^+$.

(c) Suppose that G is the Gamma(r, θ) distribution with density

$$g_{r,\theta}(v) = \frac{\theta^r v^{r-1}}{\Gamma(r)} \exp(-\theta v) 1_{(0,\infty)}(v).$$

Compute $E(W)$ and $Var(W)$ in this case by arguing conditionally. Hint: Use problem 3 above and recall that for $\Lambda \sim$ Gamma(r, θ), $E\Lambda = r/\theta$ and $Var(\Lambda) = r/\theta^2$.

(d) In the same special case as in (c), provide an easy further bound for the bound in (b) when $\lambda \equiv E\Lambda$. (Is there a name for the (unconditional) distribution of W in

this case?)

5. (35 points). Suppose that $X_n \sim$ Poisson(n); thus $X_n \stackrel{d}{=} \sum_1^n Y_i$ where Y_i are independent Poisson(1) random variables.

(a) Compute

$$E\left\{\left(\frac{X_n - n}{\sqrt{n}}\right)^-\right\}$$

explicitly and thereby show that this expectation equals $n^{n+1}e^{-n}/(\sqrt{n}n!)$. (Recall that $Y^- = -Y1_{[Y\leq 0]}$.)

(b) Show that $(X_n - n)/\sqrt{n} \to_d Z \sim N(0,1)$. (You may appeal to one of our theorems.)

(c) Use (b) and a uniform integrability argument to show that $E[(X_n - n)^-/\sqrt{n}] \to E[Z^-]$.

(d) Compute $E[Z^-]$.

(e) Combine (a) - (d) to show that $n! \sim \sqrt{2\pi n}(n/e)^n$; i.e. $n!/(\sqrt{2\pi n}(n/e)^n) \to 1$.