Statistics 522, Problem Set 9 Solutions

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1. Exercise 18.3.5, PfS page 477. Let $\{X_n, \mathcal{A}_n\}_{n=0}^{\infty}$ be a sub-martingale with $X_n \ge 0$. Let $r > 1$. Then $\{X_n^r\}$ is uniformly integrable if and only if $\{X_n^r\}$ is integrable.

   Solution: Uniform integrability implies integrability, so it remains only to prove the reverse implication. Suppose that $\{X_n^r\}$ is integrable. Then $\{X_n\}$ is uniformly integrable, and hence by the s-martingale convergence theorem 18.3.1, $X_n \to X_\infty \in L_1$ where $\{X_n, \mathcal{A}_n\}_{n=0}^{\overline{\infty}}$ is a sub-mg; i.e. $E(X_\infty|\mathcal{A}_n) \ge X_n$ a.s. and

   $$E(X_\infty^r) = E(\liminf X_n^r) \le \liminf E(X_n^r) \le \sup_n E(X_n^r) < \infty$$

   by Fatou's lemma and integrability of $\{X_n^r\}$. Hence by the conditional Jensen inequality,

   $$E(X_n^r) \le E\{E(X_\infty|\mathcal{A}_n)^r\} \le E\{E(X_\infty^r|\mathcal{A}_n)\} = E(X_\infty^r)$$

   and it follows from Vitali's theorem that $\{X_n^r\}$ is uniformly integrable.

   Alternatively, by Doob's L_r−maximal inequality, since $\{X_n, \mathcal{A}_n\}$ is a sub-martingale,

   $$E\left\{\left(\max_{1\le k\le n} X_k\right)^r\right\} \le \left(\frac{r}{r-1}\right)^r E|X_n|^r,$$

   and hence, by the monotone convergence theorem,

   $$E[\sup_{1\le k<\infty} X_k^r] \le \left(\frac{r}{r-1}\right)^r \sup_n E|X_n|^r < \infty .$$

   Thus with $Y \equiv \sup_{1\le k<\infty} X_k$, it follows that

   $$\sup_n E\left\{X_n^r 1_{[X_n^r \ge \lambda]}\right\} \le E(Y^r 1_{[Y^r\ge\lambda]} \to 0$$

   as $\lambda \to \infty$; i.e. $\{X_n^r\}$ is uniformly integrable.

2. Exercise 18.3.7, PfS page 477. Let $r > 1$. Let $\{X_n, \mathcal{A}_n\}_{n=0}^{\infty}$ be a martingale. Then the following are equivalent:
   (10) The $|X_n|^r$−process is integrable.
   (11) $X_n \to_r X_\infty$
   (12) The X_n's are uniformly integrable (thus $X_n \to$ (some X_∞) a.s.) and $X_\infty \in L_r$.
   (13) The $|X_n|^r$'s are uniformly integrable.
   (14) $\{|X_n|^r, \mathcal{A}_n\}_{n=0}^{\overline{\infty}}$ is a submg and $E|X_n|^r \nearrow E|X_\infty|^r$.

Solution: Suppose that (10) holds. Then $|X_n|^r$ is an integrable sub-mg. Thus the $|X_n|^r$ are uniformly integrable by the preceding problem. Thus (13) holds.
Suppose (13) holds. Then $\{X_n\}$ is uniformly integrable, and $X_n \to_{a.s.} X_\infty \in L_1$ and

$$E|X_\infty|^r = E(\liminf |X_n|^r) \leq \liminf E|X_n|^r \leq \sup_n E|X_n|^r < \infty\,,$$

so $X_\infty \in L_r$; i.e. (12) holds.
Suppose (12) holds. Then $\{|X_n|, \mathcal{A}_n\}_{n=0}^{\infty}$ is a sub-martingale by Theorem 16.3.1. Thus $|X_n| \leq E(|X_\infty|\mathcal{A}_n)$, so $|X_n|^r \leq \{E(|X_\infty||\mathcal{A}_n)\}^r \leq E(|X_n|^r|\mathcal{A}_n)$ a.s., and hence $E|X_n|^r \leq E|X_\infty|^r < \infty$; i.e. (10) holds.
Thus (10) iff (12) iff (13) holds.
Now (11) implies (10) since

$$E|X_n|^r \leq c_r\{E|X_n - X_\infty|^r + E|X_\infty|^r\}$$

by the c_r-inequality.
Suppose that (13) holds. Then $X_n \to_{a.s.} X_\infty \in L_r$ (by (13) implies (12)), and since $\{|X_n|^r, \mathcal{A}_n\}_{n=0}^{\infty}$ is a sub-mg,

$$\limsup_{n\to\infty} E|X_n|^r \leq E|X_\infty|^r < \infty\,.$$

Hence $X_n \to_r X_\infty$ by Vitali's theorem; i.e. (11) holds. Thus (10) iff (12) iff (13) iff (14).

3. Let $X_1, X_2, \ldots$ be i.i.d rv's with $P(X = 1) = p$, $P(X = -1) = 1 - p \equiv q$, where $0 < p < 1$ and $p \neq q$. Suppose that a, b are integers with $-a < 0 < b$. Define

$$S_n = X_1 + \ldots + X_n, \qquad T \equiv \inf\{n : S_n = -a, \text{or} \;\; S_n = b\}\,.$$

Let $\mathcal{F}_n \equiv \sigma[X_1, \ldots, X_n]$, $\mathcal{F}_0 = \{\emptyset, \Omega\}$. Prove that $M_n \equiv (q/p)^{S_n}$ and $N_n = S_n - n(p-q)$ define martingales M_n and N_n. How would you use these martingales to deduce the values of $P(S_T = -a)$ and $E(S_T)$? [Hint: see PfS, pages 499-500.]

Solution: T is clearly a stopping time and, for each n

$$P(T \leq n + b|\mathcal{F}_n) \geq p^{b-S_n} + q^{S_n} \geq (p \wedge q)^b \equiv \epsilon > 0$$

since $p \in (0, 1)$.

Thus the hypotheses of Williams PwM, E10.5, page holds with $N = b$ and $\epsilon \equiv (p \wedge q)^b$. Thus $E(T) < \infty$, and the third set of sufficient conditons for Doob's optional sampling theorem hold. Since $\{S_n - n(p-q), \mathcal{F}_n\}$ and $\{(q/p)^{S_n}, \mathcal{F}_n\}$ are both martingales, we conclude from Doob's optional sampling theorem that

$$E\left(\frac{q}{p}\right)^{S_T} = E\left(\frac{q}{p}\right)^{S_0} = \left(\frac{q}{p}\right)^{0} = 1\,. \tag{0.1}$$

But the left side of (0.1) equals

$$\left(\frac{q}{p}\right)^b P(S_T = b) + \left(\frac{q}{p}\right)^{-a} P(S_T = -a) \equiv \left(\frac{q}{p}\right)^b p_b + \left(\frac{q}{p}\right)^{-a} (1 - p_b) .$$

Thus we can solve for p_b to obtain

$$p_b = P(S_T = b) = \frac{1 - (q/p)^{-a}}{(q/p)^b - (q/p)^{-a}} = \frac{(q/p)^a - 1}{(q/p)^{a+b} - 1},$$

and

$$p_a = P(S_T = -a) = 1 - p_b = \frac{(q/p)^b - 1}{(q/p)^b - (q/p)^{-a}} = \frac{1 - (p/q)^b}{1 - (p/q)^{a+b}} .$$

It follows that

$$\begin{aligned} E(S_T) &= bp_b + (-a)p_a \\ &= b\left(1 - \frac{1 - (p/q)^b}{1 - (p/q)^{a+b}}\right) - a\frac{1 - (p/q)^b}{1 - (p/q)^{a+b}} \\ &= b - (a+b)\frac{1 - (p/q)^b}{1 - (p/q)^{a+b}} . \end{aligned}$$

Since $\{S_n - n(p-q)\} = \{S_n - n\mu\}$ is a martingale, we deduce that $E(S_T - T\mu) = 0$ and hence that

$$E(T) = \frac{1}{\mu}E(S_T) = \frac{1}{\mu}\left\{b - (a+b)\frac{1 - (p/q)^b}{1 - (p/q)^{a+b}}\right\} .$$

You should also take a look at the situation for $p = q = 1/2$ in Section 18.7, page 499.

4. Exercise 18.7.2, PfS page 500. Suppose tht S_μ is Brownian motion with drift: $S_\mu(t) = S(t) + \mu t$ for $t \ge 0$. Let $\tau_{ab} \equiv \tau \equiv \inf\{t \ge 0 : \; S_\mu(t) = -a \text{ or } b\}$ where $-a < 0 < b$.
Claim 1: $S_0(t)$, $S_0^2(t) - t$, $S_\mu(t) - \mu t$ are mean 0 martingales, and, with $\theta = -2\mu$,

$$\exp(\theta[S_\mu(t) - \mu t] - \theta^2 t/2) = \exp(-2\mu[S(t) + \mu t])$$

is a mean 1 martingale.
Claim 2: If $\mu = 0$, $P(S(\tau) = -a) = b/(a+b)$ and $E\tau = ab$.
Claim 3: If $\mu \neq 0$, then

$$P(S(\tau) = -a) = \frac{1 - e^{2\mu b}}{1 - e^{2\mu(a+b)}}$$

and

$$E(\tau) = \frac{b}{\mu} - \frac{a+b}{\mu}\frac{1 - e^{2\mu b}}{1 - e^{2\mu(a+b)}} .$$

Claim 4: If $\mu < 0$, then $P(\|S_\mu\|_0^\infty \geq b) = \exp(-2|\mu|b)$ for all $b > 0$; i.e. $\|S_\mu\|_0^\infty \sim$ Exponential$(2|\mu|)$.
(Note the analogies with problem 3.)

Suppose tht S_μ is Brownian motion with drift: $S_\mu(t) = S(t) + \mu t$ for $t \geq 0$. Let $\tau_{ab} \equiv \tau \equiv \inf\{t \geq 0 : \; S_\mu(t) = -a \text{ or } b\}$ where $-a < 0 < b$.
Claim 1: $S_0(t)$, $S_0^2(t) - t$, $S_\mu(t) - \mu t$ are mean 0 martingales, and, with $\theta = -2\mu$,

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is a mean 1 martingale.
Proof of claim 1: Since standard Brownian motion S has independent increments, with $\mathcal{A}_t \equiv \sigma[S(s), 0 \leq s \leq t]$ we have, for $0 \leq s \leq t$,

$$\begin{aligned} E(S(t)|\mathcal{A}_s) &= E(S(t) - S(s) + S(s)|\mathcal{A}_s) \\ &= E(S(t) - S(s)|\mathcal{A}_s) + E(S(s)|\mathcal{A}_s) \\ &= E(S(t) - S(s)) + S(s) = 0 + S(s) = S(s) \qquad \text{a.s.} \end{aligned}$$

so that $\{S(t), \mathcal{A}_t\}_{t\geq 0}^\infty$ is a zero - mean martingale. Since $S_\mu(t) - \mu t = S_0(t) = S(t)$, it follows immediately that $\{S_\mu(t) - \mu t, \mathcal{A}_t\}_{t\geq 0}^\infty$ is also a 0-mean martingale. To see that $\{S^2(t) - t, \mathcal{A}_t\}_{t\geq 0}^\infty$ is a zero-mean martingale, we calculate

$$\begin{aligned} E(S^2(t) - t|\mathcal{A}_s) &= E([S(t) - S(s) + S(s)]^2 - (t - s + s)|\mathcal{A}_s) \\ &= E((S(t) - S(s))^2 - (t - s)|\mathcal{A}_s) \\ &\quad\; + E(2(S(t) - S(s))S(s)|\mathcal{A}_s) \\ &\quad\; + E(S^2(s) - s|\mathcal{A}_s) \\ &= E(S(t) - S(s)^2) - (t - s) + 2S(s)E(S(t) - S(s)) \\ &\quad\; + (S^2(s) - s) \\ &= 0 + 0 + S^2(s) - s = S^2(s) - s \qquad \text{a.s.} \end{aligned}$$

so that the claim holds. (Note that this shows that $\langle S \rangle(t) = t$ is the predictable variation process corresponding to the sub - martingale $S^2(t)$.) To see that $Y_t = \exp(\theta[S_\mu(t) - \mu t] - \theta^2 t/2) = \exp(-2\mu[S(t) + \mu t])$ is a mean 1 martingale, note that $Y_t = \exp(\theta S(t) - \theta^2 t/2)$ and hence

$$\begin{aligned} E(Y_t|\mathcal{A}_s) &= E(\exp(\theta(S(t) - S(s)))|\mathcal{A}_s) \cdot E(\exp(\theta S(s) - \theta^2 s/2)|\mathcal{A}_s) \\ &\qquad \cdot \exp(\theta^2(s/2 - t/2)) \\ &= E(\exp(\theta(S(t) - S(s)))) \cdot \exp(\theta^2(s/2 - t/2)) \cdot Y_s \qquad \text{a.s.} \\ &= \exp(\theta^2(t/2 - s/2)) \cdot \exp(\theta^2(s/2 - t/2)) \cdot Y_s \qquad \text{a.s.} \\ &= Y_s\,, \end{aligned}$$

so that Y_t is a mean 1 mg. The second part of this holds simply because, with $\theta = -2\mu$ we have

$$\theta[S_\mu - \mu t] - \theta^2 t/2 = -2\mu S_\mu + 2\mu^2 t - 4\mu^2 t/2 = -2\mu S_\mu(t)\,.$$

Claim 2: If $\mu = 0$, $P(S(\tau) = -a) = b/(a+b)$ and $E\tau = ab$.
Claim 3: If $\mu \neq 0$, then

$$P(S(\tau) = -a) = \frac{1 - e^{2\mu b}}{1 - e^{2\mu(a+b)}}$$

and

$$E(\tau) = \frac{b}{\mu} - \frac{a+b}{\mu}\frac{1 - e^{2\mu b}}{1 - e^{2\mu(a+b)}}\,.$$

To prove claim 2, first consider the bounded stopping times $\tau \wedge k$. Then by the basic optional sampling theorem,

$$0 = E(S_0^2(\tau \wedge k) - \tau \wedge k)\,. \tag{0.2}$$

Now $\tau \wedge k \nearrow \tau$, so that $E(\tau \wedge k) \to E(\tau)$ by the monotone convergence theorem, while $S_0(\tau \wedge k) \to S_0(\tau)$ with $|S_0(\tau \wedge k)| \leq a \vee b < \infty$ for all k, and hence $E(S_0^2(\tau \wedge k)) \to E(S_0^2(\tau))$ by the dominated convergence theorem. Thus taking limits across (0.2) yields

$$E(S_0^2(\tau)) = E(\tau)\,,$$

and when $\mu = 0$, this implies that $E(\tau) < \infty$. By playing this game with the martingale S, we find that $E(S(\tau \wedge k)) = 0$, and by the dominated convergence theorem, $E(S(\tau)) = 0$. Since $S(\tau)$ takes on the two values $-a$ and b, we have

$$0 = E_0 S(\tau) = -aP_0(S(\tau) = -a) + bP_0(S(\tau) = b) = -a(1 - p_b) + bp_b$$

so that $p_b = a/(b+a)$, $p_a = 1 - p_b = b/(b+a)$. From $E(S_0^2(\tau)) = E(\tau)$ it then follows that

$$E(\tau) = a^2 p_a + b^2 p_b = a^2 \frac{b}{b+a} + b^2 \frac{a}{b+a} = ab\,,$$

completing the proof of Claim 2.
Proof of Claim 3. Similarly, when $\mu \neq 0$, the basic optional sampling theorem yields

$$0 = E(S_\mu(\tau \wedge k) - (\tau \wedge k)\mu)\,. \tag{0.3}$$

Now $\tau \wedge k \nearrow \tau$, so that $E(\tau \wedge k) \to E(\tau)$ by the monotone convergence theorem, while $S_\mu(\tau \wedge k) \to S_\mu(\tau)$ with $|S_\mu(\tau \wedge k)| \leq a \vee b < \infty$ for all k, and hence $E(S_\mu(\tau \wedge k)) \to E(S_\mu(\tau))$ by the dominated convergence theorem. Thus taking limits across (0.3) yields

$$E(S_\mu(\tau)) = \mu E(\tau)\,,$$

and this implies that $E(\tau) < \infty$ for $\mu \neq 0$. Again the basic optional sampling theorem implies that

$$E(Y(0)) = 1 = E\exp(-2\mu S_\mu(\tau \wedge k))\,,$$

for each k, and by the dominated convergence theorem this yields

$$\begin{aligned} E(Y(0)) = 1 &= E\exp(-2\mu S_\mu(\tau)) \\ &= P(S_\mu(\tau) = -a)\exp(2\mu a) + P(S_\mu(\tau) = b)\exp(-2\mu b) \\ &= p_a \exp(2\mu a) + (1 - p_a)\exp(-2\mu b) \\ &= p_a(\exp(2\mu a) - \exp(-2\mu b)) + \exp(-2\mu b) \end{aligned}$$

so that

$$p_a = \frac{1 - \exp(-2\mu b)}{\exp(2\mu a) - \exp(-2\mu b)} = \frac{1 - \exp(2\mu b)}{1 - \exp(2\mu(a+b))}\,.$$

Then, finally, since $E(S_\mu(\tau)) = \mu$,

$$\begin{aligned} E(\tau) &= \frac{E(S_\mu(\tau))}{\mu} \\ &= \frac{1}{\mu}\{-ap_a + b(1 - p_a)\} \\ &= \frac{1}{\mu}\left\{b - (a+b)\frac{1 - \exp(2\mu b)}{1 - \exp(2\mu(a+b))}\right\}\,. \end{aligned}$$

Note that when $\mu < 0$ we have

$$\begin{aligned} P(\|S_\mu\|_0^\infty \geq b) &= \lim_{a\to\infty} P(\tau_{ab} < \infty) \\ &= \lim_{a\to\infty} P(S_\mu(\tau_{ab}) = b) \\ &= \exp(-2|\mu|b) \end{aligned}$$

so that $\|S_\mu\|_0^\infty \sim \text{Exponential}(2|\mu|)$.

5. Suppose that $X_1, X_2, \ldots$ are independent random variables on $(\Omega, \mathcal{A})$ and that X_n has density p_n or q_n under P or Q respectively where p_n and q_n are (for simplicity) everywhere positive on $\mathbb{R}$. Let $\mathcal{F} = \sigma[X_1, X_2, \ldots]$ and $\mathcal{F}_n = \sigma[X_1, \ldots, X_n]$ for $n \geq 1$. Let $Y_n \equiv q_n(X_n)/p_n(X_n)$.
(a) Show that

$$M_n \equiv \left.\frac{dQ}{dP}\right|_{\mathcal{F}_n} = Y_1 \cdots Y_n$$

where the Y_n's are independent and have mean 1 under P; Hence the likelihood ratio martingale of Example 1.14 is the Kakutani product martingale of Example 1.15.
(b) Show that Q is absolutely continuous relative to P on $\mathcal{F}$ if and only if the martingale $\{M_n, \mathcal{F}_n\}$ is uniformly integrable.
(c) Conclude from Kakutani's theorem (PfS Example 4.4, pages 482-483) that $Q \ll P$ on $\mathcal{F}$ if and only if

$$\prod_{n=1}^\infty E(Y_n^{1/2}) = \prod_{n=1}^\infty \int_{\mathbb{R}} \sqrt{p_n(x)q_n(x)}dx > 0\,.$$

(d) Construct two examples of sequences p_n and q_n, one in which the condition in (c) holds and one in which it fails. What is the statistical meaning when it holds and when it fails?

Solution: (a) Let $A_i \in \sigma(X_i)$ for $i = 1, \dots, n$. Then

$$\begin{aligned}
E_P\{1_{A_1\times\cdots\times A_n}\frac{dQ}{dP}\} &= E_Q\{1_{A_1\times\cdots\times A_n}\} \\
&\qquad \text{by definition of the Radon-Nikodym derivative} \\
&= \prod_{i=1}^n E_Q(1_{A_i}) \qquad \text{by independence} \\
&= \prod_{i=1}^n \int_{\mathbb{R}} 1_{A_i}(x) q_i(x) d\mu(x) \qquad \text{by existence of the densities } q_i \\
&= \prod_{i=1}^n \int_{\mathbb{R}} 1_{A_i}(x) \frac{q_i(x)}{p_i(x)} p_i(x) d\mu(x) \\
&= \prod_{i=1}^n E_P\{1_{A_i} Y_i\} \\
&= E_P\{1_{A_1\times\cdots\times A_k} Y_1 \cdots Y_n\} \qquad \text{by independence}\,.
\end{aligned}$$

Now $Y_1 \cdots Y_n$ is $\mathcal{F}_n$ measurable (since it is a function of $X_1, \dots, X_n$ and agrees with $dQ/dP|_{\mathcal{F}_n}$ on the $\overline{\pi}-$ system $\sigma(X_1) \times \cdots \times \sigma(X_n)$. Thus the claimed equality holds. The Y_i's are independent because the X_i's are independent and they have mean 1 because

$$E_P Y_i = \int_{\mathbb{R}} \frac{q_i(x)}{p_i(x)} p_i(x) d\mu(x) = \int_{\mathbb{R}} q_i(x) dx = 1\,.$$

(b) If $Q \ll P$, with Radon-Nikodym derivative $dQ/dP \equiv Z$, then $M_n = E(Z|\mathcal{F}_n)$ with $E_P(Z) = Q(\mathbb{R}^\infty) = 1$, so $\{M_n, \mathcal{F}_n\}_{n=0}^{\overline{\infty}}$ is a martingale closed at infinity and is uniformly integrable. Conversely, if $\{M_n\}$ is uniformly integrable, then $M_n \to_{a.s.} M_\infty$ and $E(M_\infty|\mathcal{F}_n) = M_n$ almost surely for every n. Now consider the measures Q and $\tilde{Q}$ defined by

$$\tilde{Q}(A) = E\{1_A M_\infty\}\,.$$

These measures agree on the π-system $\cup\mathcal{F}_n$, and hence they agree on $\mathcal{F} = \sigma(\cup\mathcal{F}_n)$. This implies (by the pi-lambda theorem) that Q and $\tilde{Q}$ agree on $\mathcal{F}$, and hence $M_\infty = dQ/dP$ on $\mathcal{F}$, and $Q \ll P$.

(c) By Kakutani's theorem we conclude that Q is absolutely continuous with respect to P on $\mathcal{F}$ if and only if

$$\prod_{n=1}^\infty E(Y_n^{1/2}) = \prod_{n=1}^\infty \int_{\mathbb{R}} (q_n(x)/p_n(x))^{1/2} p_n(x) dx = \prod_{n=1}^\infty \int_{\mathbb{R}} \sqrt{p_n(x) q_n(x)} dx > 0\,.$$

Since this is symmetric in p_n and q_n and since these densities are everywhere positive, we also can conclude that P is absolutely continuous with respect to Q on $\mathcal{F}$; thus Q and P are mutually absolutely continuous or *equivalent* on $\mathcal{F}$.
(d) Suppose that $p_n(x) = \exp(-x)1_{[0,\infty)}(x)$ and $q_n(x) = \lambda_n \exp(-\lambda_n x)1_{[0,\infty)}(x)$ with $\lambda_n = 1 + c_n$ where $c_n \to 0$. Then we compute

$$E(Y_n^{1/2}) = \int_0^\infty \lambda_n^{1/2} \exp(-(1+\lambda_n)x/2)dx = \frac{2\lambda_n^{1/2}}{1+\lambda_n},$$

and

$$\begin{aligned} H^2(P_n, Q_n) &= \frac{1}{2}\int (\sqrt{p_n(x)} - \sqrt{q_n(x)})^2 dx = 1 - E(Y_n^{1/2}) \\ &= 1 - \frac{2\lambda_n^{1/2}}{1+\lambda_n} \\ &= \frac{1+\lambda_n - 2\lambda_n^{1/2}}{1+\lambda_n} \\ &= \frac{2+c_n - 2(1+c_n)^{1/2}}{2+c_n} \\ &\sim \frac{1}{8}c_n^2 \qquad \text{as } n \to \infty \end{aligned}$$

since $c_n \to 0$ and $(1+c_n)^{1/2} = 1 + (1/2)c_n - (1/8 + o(1))c_n^2$. Thus if $c_n = n^{-r}$ with $r > 1/2$ it follows that

$$\sum_1^\infty (1 - E(Y_n^{1/2})) = \sum_1^\infty H^2(P_n, Q_n) < \infty,$$

and $Q \ll P$ on $\mathcal{F}$. If $c_n = n^{-1/2}$, then

$$\sum_1^\infty (1 - E(Y_n^{1/2})) = \sum_1^\infty H^2(P_n, Q_n) = \infty,$$

and by Kakutuni's theorem we conclude that $M_\infty = 0$ almost surely P. In this case Q and P are singular on $\mathbb{R}^\infty$: there is a set $A \subset \mathbb{R}^\infty$ such that $Q(A) = 1$ and $P(A) = 0$; i.e. $P(A^c) = 1$.