Statistics 522, Problem Set 8 Solutions

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1. PfS, Exercise 18.1.6, page 471.

Solution: In example 1.12, $N(t)$ is a Poisson process with intensity λ; and $M(t) = N(t) - \lambda t$ and $M^2(t) - \lambda t$ are both martingales. A natural exponential martingale to consider is

$$Y(t) \equiv Y_c(t) \equiv \frac{\exp(cM(t))}{E\exp(cM(t))}\,.$$

Since

$$\begin{aligned} E\exp(cM(t)) &= E\exp(c(N(t)-\lambda t)) \\ &= \exp(-c\lambda t)E\exp(c(N(t)) \\ &= \exp(-c\lambda t)\exp((e^c-1)\lambda t)\,, \end{aligned}$$

we find that

$$Y(t) = \exp(cN(t) - (e^c-1)\lambda t)\,.$$

I claim that $\{Y(t), \mathcal{A}_t\}_{t=0}^{\infty}$ is a martingale on $[0,\infty)$. To see this, note that for $0 \le s < t < \infty$ we have

$$\begin{aligned} E\left(Y(t)|\mathcal{A}_s\right) &= E\left(\exp(cN(t)-(e^c-1)\lambda t)|\mathcal{A}_s\right) \\ &= E\left(\exp(c(N(t)-N(s)) - (e^c-1)\lambda(t-s))\exp(cN(s)-(e^c-1)\lambda s)|\mathcal{A}_s\right) \\ &= Y(s)E\left(\exp(c(N(t)-N(s)) - (e^c-1)\lambda(t-s))|\mathcal{A}_s\right) \qquad \text{a.s.} \\ &= Y(s)E\left(\exp(c(N(t)-N(s)) - (e^c-1)\lambda(t-s))\right) \qquad \text{a.s.} \\ &\qquad \text{since } \; N(t)-N(s) \; \text{ is independent of } \; \mathcal{A}_s \\ &= Y(s)\cdot 1 = Y(s) \qquad \text{a.s.}\,, \end{aligned}$$

and hence $\{Y(t), \mathcal{A}_t\}_{t=0}^{\infty}$ is a martingale. Note that

$$\begin{aligned} Y_c'(t) \equiv \frac{d}{dc}Y_c(t)|_{c=0} &= Y_c(t)\left(N(t) - e^c\lambda t\right)|_{c=0} \\ &= Y_0(t)\left(N(t)-e^c\lambda t\right)|_{c=0} \\ &= N(t) - \lambda t = M(t)\,, \end{aligned}$$

while

$$\begin{aligned} Y_c''(t) \equiv \frac{d^2}{dc^2}Y_c(t)|_{c=0} &= Y_c(t)\left(N(t)-e^c\lambda t\right)^2|_{c=0} + Y_c(t)\left(-e^c\lambda t\right)|_{c=0} \\ &= Y_0(t)\left(N(t)-e^c\lambda t\right)|_{c=0} \\ &= M^2(t) - \lambda t\,. \end{aligned}$$

2. PfS, Exercise 18.3.3, page 477. (a) Let $\cdots \subset \mathcal{A}_{-1} \subset \mathcal{A}_0 \subset \cdots$ be sub $\sigma-$fields of $\mathcal{A}$. Let $X \in \mathcal{L}_1(\Omega, \mathcal{A}, P)$. Let $Y_n \equiv E(X|\mathcal{A}_n)$. Then $(Y_n, \mathcal{A}_n)_{n=-\infty}^{\infty}$ is a uniformly integrable martingale.
(b) Let $Y_t = E(Y|\mathcal{D}_t)$ for an arbitrary collection of sub sigma fields $\mathcal{D}_t$ of $\mathcal{A}$. These Y_t's are necessarily uniformly integrable.

Solution: (a) Now

$$P(|Y_n| \geq \lambda) \leq \frac{E|Y_n|}{\lambda} = \frac{E|E(X|\mathcal{A}_n)|}{\lambda} \leq \frac{E|X|}{\lambda}$$

so that

$$\sup_{-\infty<n<\infty} P(|Y_n| \geq \lambda) \leq \frac{E|X|}{\lambda} \to 0$$

as $\lambda \to \infty$. Thus we have

$$\begin{aligned} E\{|Y_n|1_{[|Y_n|\geq\lambda]}\} &= E\{|E(X|\mathcal{A}_n)|1_{[|Y_n|\geq\lambda]}\} \\ &\leq E\{E(|X||\mathcal{A}_n\}1_{[|Y_n|\geq\lambda]}\} \\ &\qquad \text{by the conditional Jensen inequality} \\ &= E\{E(|X|1_{[|Y_n|\geq\lambda]}|\mathcal{A}_n)\} \\ &\qquad \text{since } Y_n \text{ is } \mathcal{A}_n \text{ measurable} \\ &= E\{|X|1_{[|Y_n|\geq\lambda]}\} \to 0 \end{aligned}$$

uniformly in n by the absolute continuity of the integral. Thus $(Y_n, \mathcal{A}_n)_{n=-\infty}^{\infty}$ is a uniformly integrable mg.
(a) Similarly,

$$P(|Y_t| \geq \lambda) \leq \frac{E|Y_t|}{\lambda} = \frac{E|E(Y|\mathcal{D}_t)|}{\lambda} \leq \frac{E|Y|}{\lambda}$$

so that

$$\sup_{t\in T} P(|Y_t| \geq \lambda) \leq \frac{E|Y|}{\lambda} \to 0$$

as $\lambda \to \infty$. Thus we have

$$E\{|Y_t|1_{[|Y_t|\geq\lambda]}\} = E\{|E(Y|\mathcal{D}_t)|1_{[|Y_t|\geq\lambda]}\} \leq E\{|Y|1_{[|Y_t|\geq\lambda]}\} \to 0$$

uniformly in $t \in T$ by the absolute continuity of the integral. Thus $\{Y_t : \ t \in T\}$ is uniformly integrable.

3. Suppose that S and T are stopping times relative to the filtration $\{\mathcal{F}_n\}$. Show that $S \wedge T$, $S \vee T$, and $S + T$ are also stopping times.

Solution: To see that $S \wedge T$ is a stopping time, note that

$$[S \wedge T > n] = [S > n] \cap [T > n] = [S \leq n]^c \cap [T \leq n]^c$$

where $[S \le n] \in \mathcal{F}_n$ and $[T \le n] \in \mathcal{F}_n$ since S and T are stopping times. It follows that $[S \wedge T > n] \in \mathcal{F}_n$, and hence also $[S \wedge T \le n] \in \mathcal{F}_n$; i.e. $S \wedge T$ is a stopping time.
To see that $S \vee T$ is a stopping time, note that

$$[S \vee T \le n] = [S \le n] \cap [T \le n]$$

where $[S \le n] \in \mathcal{F}_n$ and $[T \le n] \in \mathcal{F}_n$. It follows that $[S \vee T \le n] \in \mathcal{F}_n$; i.e. $S \vee T$ is a stopping time.
To see that $S + T$ is a stopping time, note that

$$[S + T \le n] = \cup_{k=0}^{n} [S = k] \cap [T = n - k]$$

where $[S = k] \in \mathcal{F}_k \subset \mathcal{F}_n$ and where $[T = n-k] \in \mathcal{F}_{n-k} \subset \mathcal{F}_n$, for $k = 0, \ldots, n$. Hence $[S + T \le n] \subset \mathcal{F}_n$; i.e. $S + T$ is a stopping time.

4. Polyá's urn: At time 0, an urn contains 1 black ball and 1 white ball. At each time $1, 2, 3, \ldots$, a ball is chosen at random from the urn, and is replaced together with a new ball of the same color. Just after time n, there are therefore $n + 2$ balls in the urn, of which $B_n + 1$ are black, where B_n is the number of black balls chosen by time n. Let $M_n = (B_n + 1)/(n + 2)$, the proportion of black balls in the urn just after time n. Prove that (relative to a natural filtration which you should specify) M_n is a martingale. Prove that $P(B_n = k) = 1/(n+1)$ for $0 \le k \le n$. What is the distribution of $\Theta \equiv \lim_n M_n$? Prove that for $0 < \theta < 1$,

$$N_n^\theta \equiv \frac{(n+1)!}{B_n!(n - B_n)!} \theta^{B_n} (1-\theta)^{n - B_n}$$

defines a martingale N_n^θ.

Solution: Let $\mathcal{F}_n \equiv \sigma(B_1, \ldots, B_n)$. Note that $M_n \equiv (B_n + 1)/(n+2)$ is the conditional (given $\mathcal{F}_n$) probability of drawing a black ball at the $n + 1$st draw. Thus we compute

$$\begin{aligned}
E(M_{n+1}|\mathcal{F}_n) &= E\left(\frac{B_{n+1}+1}{n+3}|\mathcal{F}_n\right) = \frac{1}{n+3} E(B_{n+1} + 1|\mathcal{F}_n) \\
&= \frac{1}{n+3}\{(B_n + 1)(1 - M_n) + (B_n + 2)M_n\} \\
&= \frac{1}{n+3}\{B_n + 1 - M_n + 2M_n\} \\
&= \frac{1}{n+3}\{(n+2)M_n + M_n\} = M_n \qquad \text{a.s.}.
\end{aligned}$$

Hence $\{M_n, \mathcal{F}_n\}$ is a martingale. Similarly, letting

$$p_n(k) \equiv \frac{(n+1)!}{k!(n-k)!} \theta^k (1-\theta)^{n-k},$$

the process $N_n^\theta = p_n(B_n)$ and

$$\begin{aligned}
E(N_{n+1}^\theta|\mathcal{F}_n) &= E(p_{n+1}(B_{n+1})|\mathcal{F}_n) \\
&= p_{n+1}(B_n)(1-M_n) + p_{n+1}(B_n+1)M_n \\
&= \frac{(n+2)!}{B_n!(n+1-B_n)!}\theta^{B_n}(1-\theta)^{n+1-B_n}\frac{(n+1-B_n)}{(n+2)} \\
&\quad + \frac{(n+2)!}{(B_n+1)!(n+1-B_n-1)!}\theta^{B_n+1}(1-\theta)^{n+1-B_n-1}\frac{(B_n+1)}{(n+2)} \\
&= \frac{(n+1)!}{B_n!(n-B_n)!}\theta^{B_n}(1-\theta)^{n-B_n}\{(1-\theta)+\theta\} \\
&= p_n(B_n) \equiv N_n^\theta \qquad \text{a.s.},
\end{aligned}$$

so $\{N_n^\theta, \mathcal{F}_n\}$ is a martingale. This implies that $EN_n^\theta = EN_0^\theta = 1$ for all $\theta \in (0,1)$, or

$$E\left\{\frac{n!}{B_n!(n-B_n)!}\theta^{B_n}(1-\theta)^{n-B_n}\right\} = \frac{1}{n+1}\,. \tag{0.1}$$

This equality clearly holds if $P(B_n = k) = 1/(n+1)$ for $k = 0, \ldots, n$. On the other hand, (1.1) implies, by letting $\alpha = \theta/(1-\theta)$, that, with $p_k = P(B_n = k)$,

$$\sum_{k=0}^n \frac{n!}{k!(n-k)!}\alpha^k p_k = \frac{1}{n+1}(1+\alpha)^n = \frac{1}{n+1}\sum_{k=0}^n \frac{n!}{k!(n-k)!}\alpha^k\,,$$

and this yields $p_k = 1/(n+1)$ by matching coefficients.

The distribution of B_n is a discrete uniform distribution on $0, \ldots, n$ for every n, so the distribution of M_n is a discrete uniform distribution on $0 < 1/(n+1) < \cdots < (n+1)/(n+2) < 1$ and it is clear that $M_n \to_d U(0,1)$ as $n \to \infty$; $P(M_n \le u) = [(n+2)u]/(n+1) \to u = P(U \le u)$ where $U \sim \text{Uniform}(0,1)$.

5. Let $\xi_1, \xi_2, \ldots, \xi_n$ be i.i.d. Uniform$(0,1)$ random variables, and let $\mathbb{G}_n(t) = n^{-1}\sum_1^n 1_{[0,t]}(\xi_i)$ for $0 \le t \le 1$; $\mathbb{G}_n$ is the empirical distribution function of the $\xi's$. Fix n and let $\mathcal{F}_n(t) \equiv \sigma\{\mathbb{G}_n(s) : s \le t\}$ for $0 \le t \le 1$. Consider the processes $Y_n(t) \equiv (\mathbb{G}_n(t) - t)/(1-t)$ and $Z_n(t) \equiv (1 - \mathbb{G}_n(t))/(1-t)$. Show that $(Y_n(t), \mathcal{F}_n(t) : 0 \le t < 1)$ and $(Z_n(t), \mathcal{F}_n(t) : 0 \le t < 1)$ are martingales.

Solution: Let $0 < s < t < 1$. Now $(n\mathbb{G}_n(s), n(\mathbb{G}_n(t) - \mathbb{G}_n(s)), n(1-\mathbb{G}_n(t))) \sim \text{Mult}_3(n, (s, t-s, 1-t))$, and hence by direct calculation with the mass functions the conditional distribution of $n(1-\mathbb{G}_n(t))$ given $n\mathbb{G}_n(s)$ is Binomial$(n(1-\mathbb{G}_n(s)), (1-t)/(1-s))$. Thus we compute

$$E\{n(1-\mathbb{G}_n(t))|n\mathbb{G}_n(s)\} = n(1-\mathbb{G}_n(s))\frac{1-t}{1-s} \qquad a.s..$$

Dividing across this equality by $n(1-t)$ shows that

$$E\left\{\frac{1-\mathbb{G}_n(t)}{1-t}\Big| n\mathbb{G}_n(s)\right\} = \frac{1-\mathbb{G}_n(s)}{1-s} \qquad a.s..$$

Since $\{n(1-\mathbb{G}_n(t)) : \ 0 \le t < 1\}$ is a *Markov process*,

$$E\{n(1-\mathbb{G}_n(t))|\mathcal{F}_n(s)\} = E\{n(1-\mathbb{G}_n(t)|n(1-\mathbb{G}_n(s))\} = E\{n(1-\mathbb{G}_n(t)|n\mathbb{G}_n(s)\}\,, \qquad \text{a.s.}$$

(for a proof, see below) and hence we have

$$E\left\{\frac{1-\mathbb{G}_n(t)}{1-t}\Big|\mathcal{F}_n(s)\right\} = \frac{1-\mathbb{G}_n(s)}{1-s} \qquad a.s..$$

Thus $\{Z_n(t), \mathcal{F}_n(t)\}$ is a martingale. Since $Z_n(t)$ is a mean 1 martingale, $Z_n(t)-1$ is a mean 0 martingale, and rewriting this yields

$$Z_n(t) - 1 = \frac{1-\mathbb{G}_n(t)}{1-t} - 1 = \frac{1-\mathbb{G}_n(t)-(1-t)}{1-t} = -\frac{\mathbb{G}_n(t)-t}{1-t} = -Y_n(t)\,.$$

Thus $\{Y_n(t), \mathcal{F}_n(t)\}$ is also a mean 0 martinagle.

To show that $n\mathbb{G}_n(t)$ is Markov, it suffices to show that for any fixed set $0 < t_1 < \cdots < t_k < t_{k+1} < 1$ the conditional distribution of $N_{k+1} \equiv n\mathbb{G}_n(t_{k+1})$ given $N_1 \equiv n\mathbb{G}_n(t_1), \ldots, N_k \equiv n\mathbb{G}_n(t_k)$ depends only on $N_k = n\mathbb{G}_n(t_k)$. To show this, let $D_1 = N_1$, $D_j \equiv N_j - N_{j-1}$, $j = 2, \ldots, k+1$. Then

$$\begin{aligned} & P(N_{k+1} = m | N_1 = n_1, \ldots, N_k = n_k) \\ &= P(N_{k+1} - N_k = m - n_k | D_1 = d_1, \ldots, D_k = d_k) \\ &= \frac{\frac{n!}{d_1!\cdots d_{k+1}!(n-m)!} t_1^{d_1}(t_2-t_1)^{d_2}\cdots(t_{k+1}-t_k)^{d_{k+1}}(1-t_{k+1})^{n-m}}{\frac{n!}{d_1!\cdots d_k!(n-m+d_{k+1})!} t_1^{d_1}(t_2-t_1)^{d_2}\cdots(t_k-t_{k-1})^{d_k}(1-t_k)^{n-m+d_{k+1}}} \\ &= \frac{(n-m+d_{k+1})!}{d_{k+1}!(n-m)!}\frac{(t_{k+1}-t_k)^{d_{k+1}}(1-t_{k+1})^{n-m}}{(1-t_k)^{n-m+d_{k+1}}} \\ &= P(N_{k+1} = m | N_k = n_k) \end{aligned}$$

where $n_1 \le n_2 \le \ldots \le n_k \le m \le n$, $d_1 = n_1$, $d_j = n_j - n_{j-1}$ for $j = 2, \ldots, k+1$, $d_1, \ldots, d_{k+1} \ge 0$, and $\sum_{j=1}^{k+1} d_j = m \le n$.