Statistics 522, Problem Set 7 Solutions

Wellner; 2/27/2004

1. PfS, page 161, Exercise 4.2.

   Solution: (a) When the sampling is done without replacement the joint probability distribution for (X_1, X_2) is as follows:

			X_1		
		1	2	3	
X_2	1	0	2/30	3/30	5/30
	2	2/30	2/30	6/30	10/30
	3	3/30	6/30	6/30	15/30
		5/30	10/30	15/30	1

   Hence the marginal distribution of $S = X_1 + X_2$ is given by

k	2	3	4	5	
$P(S = k)$	4/30	8/30	12/30	6/30	1

   It is easy to compute the conditional distribution of $Y = X_2$ given S (or given $\mathcal{D} = S^{-1}(\mathcal{B})$): letting $D_j = [S = j]$,

		Y		
	1	2	3	$E(Y\|\mathcal{D})$
D_3	1/2	1/2	0	3/2
D_4	3/8	2/8	3/8	2
D_5	0	1/2	1/2	5/2
D_6	0	0	1	3
$P(Y = i)$	5/30	10/30	15/30	

   Note that

   $$P(Y = i|\mathcal{D}) = \sum_{j=3}^{6} \frac{P([Y = i] \cap D_j)}{P(D_j)} 1_{D_j}$$

   satisfies $P(Y = i) = E\{P(Y = i|\mathcal{D})\}$. Also note that $E(Y) = 7/3$, and

   $$E(E(Y|\mathcal{D})) = (3/2)(4/30) + 2(8/30) + (5/2)(12/30) + 3(6/30) = 7/3\,.$$

2. PfS, page 161, Exercise 4.3.

   Solution: (a) With the events C_i as given, we compute the following probability distribution:

i	1	2	3	4
$P(Y=i)$	1/32	2/32	3/32	4/32
$P(Y=i+4)$	15/32	4/32	1/32	2/32
$P(Y=i \text{ or } i+4)$	16/32	6/32	4/32	6/32

This results in

$$E(Y) = (1\cdot 1 + 2\cdot 2 + 3\cdot 3 + 4\cdot 4 + 5\cdot 15 + 6\cdot 4 + 7\cdot 1 + 8\cdot 2)/32 = 152/32\,.$$

(b) With $\mathcal{D}$ as given we compute the conditional probability distribution and conditional expectations to be:

D_i	1	2	3	4
$P(Y=i\|\mathcal{D})$	1/16	2/6	3/4	4/6
$P(Y=i+4\|\mathcal{D})$	15/16	4/6	1/4	2/6
$E(Y\|\mathcal{D})$	76/16	28/6	16/4	32/6

This yields

$$E(E(Y|\mathcal{D})) = \frac{76}{16}\cdot\frac{16}{32} + \frac{28}{6}\cdot\frac{6}{32} + \frac{16}{4}\cdot\frac{4}{32} + \frac{32}{6}\cdot\frac{6}{32} = \frac{152}{32} = E(Y)\,.$$

(c) With $\mathcal{E}$ as given, we compute the conditional probability distribution and conditional expectations to be:

i	1	2	3	4
$P(Y=i\|\mathcal{E})$	1/22	2/22	3/10	4/10
$P(Y=i+4\|\mathcal{E})$	15/22	4/22	1/10	2/10
$P(E_j)$	22/32		10/32	

This yields

$$E(Y|\mathcal{E})(\omega) = \frac{1+4+75+24}{22} = \frac{104}{22} \qquad \omega\in E_1\,,$$

and

$$E(Y|\mathcal{E})(\omega) = \frac{9+16+7+16}{10} = \frac{48}{10} \qquad \omega\in E_2\,,$$

and hence

$$E(E(Y|\mathcal{E})) = \frac{104}{22}\frac{22}{32} + \frac{48}{10}\frac{10}{32} = \frac{152}{32} = E(Y)\,.$$

3. Suppose that $X, Y \in L_1(\Omega, \mathcal{F}, P)$ and that $E(Y|X) = X$ a.s. and $E(X|Y) = Y$ a.s.. Prove that $P(X = Y) = 1$.

Solution: Suppose first that $X, Y \in L_2(P)$. Then, by Pythagoras,

$$EX^2 = E(E(X|Y)^2) + E((X - E(X|Y))^2)\,,$$

and since $E(X|Y) = Y$ a.s. this yields

$$E(X^2) = E(Y^2) + E((X - Y)^2)\,. \tag{0.1}$$

Reversing the roles of X and Y, we also obtain, upon using $E(Y|X) = X$ a.s.,

$$E(Y^2) = E(X^2) + E((Y - X)^2)\,. \tag{0.2}$$

Adding (0.1) and (0.2) gives

$$E(X^2) + E(Y^2) = E(X^2) + E(Y^2) + 2E((X - Y)^2)\,,$$

and this implies that $E(X - Y)^2 = 0$, which in turn implies $P(X = Y) = 1$.

Now one way to proceed is to reduce the general case of $X, Y \in L_1(P)$ to the $L_2(P)$ case treated above. Instead I will prove it using the hint.

Note that

$$\begin{aligned} & E(X - Y)1_{[X>c, Y\leq c]} + E(X - Y)1_{[X\leq c, Y\leq c]} \\ = \;& E(X - Y)1_{[Y\leq c]} = E(X1_{[Y\leq c]}) - E(Y1_{[Y\leq c]}) \\ = \;& E(E(X1_{[Y\leq c]}|Y)) - E(Y1_{[Y\leq c]}) \\ = \;& E(1_{[Y\leq c]}E(X|Y)) - E(Y1_{[Y\leq c]}) \\ = \;& E(1_{[Y\leq c]}Y) - E(Y1_{[Y\leq c]}) = 0 \end{aligned} \tag{0.3}$$

using $E(X|Y) = Y$ a.s. in the last line. Similarly, reversing the roles of X and Y,

$$\begin{aligned} & E(Y - X)1_{[Y>c, X\leq c]} + E(Y - X)1_{[Y\leq c, X\leq c]} \\ = \;& E(Y - X)1_{[X\leq c]} = 0\,. \end{aligned} \tag{0.4}$$

Adding (0.3) and (0.4) yields

$$\begin{aligned} 0 \;=\;& E(X - Y)1_{[X>c, Y\leq c]} + E(X - Y)1_{[X\leq c, Y\leq c]} \\ & - E(X - Y)1_{[Y>c, X\leq c]} - E(X - Y)1_{[Y\leq c, X\leq c]} \\ =\;& E(X - Y)1_{[X>c, Y\leq c]} - E(X - Y)1_{[Y>c, X\leq c]}\,. \end{aligned}$$

Since $[X - Y > 0] = [X > Y] = \cup_{q\in\mathbf{Q}}[X > q \geq Y]$ and similarly for $[X - Y < 0]$, this yields, by summing over rationals q,

$$0 = E(X - Y)1_{[X-Y>0]} - E(X - Y)1_{[X-Y<0]} = E|X - Y|\,.$$

But this implies $P(|X - Y| = 0) = 1$, or $P(X = Y) = 1$.

4. Let $X_1, \ldots, X_n$ be i.i.d. real random variables with $E|X_1| < \infty$ and let $S_n \equiv X_1 + \cdots + X_n$. Let $\mathcal{T}_n$ be the smallest $\sigma-$algebra for which S_k are measurable for all $k \ge n$; $\mathcal{T}_n \equiv \sigma\{S_k, k \ge n\}$. Show that $E(X_j|\mathcal{T}_n) = n^{-1}S_n$ a.s. for $j = 1, \ldots, n$.

   Solution: Note that $\mathcal{T}_n = \sigma\{S_n, S_{n+1}, \ldots\} = \sigma\{S_n, X_{n+1}, X_{n+2}, \ldots\}$ and that $\sigma\{X_{n+1}, X_{n+2}, \ldots\}$ is independent of $\sigma\{X_1, S_n\}$. Thus by Theorem 8.4.1 part (23) it follows that

$$E(X_1|\mathcal{T}_n) = E(X_1|S_n)\,.$$

   But with $X_1 \sim F$, for any Borel set B we have

$$\begin{aligned} E\{1_B(S_n)X_1\} &= \int \cdots \int_{s_n \in B} x_1 dF(x_1)dF(x_2)\cdots dF(x_n) \\ &= E\{1_B(S_n)X_2\} = \cdots = E\{1_B(S_n)X_n\}\,. \end{aligned}$$

   Hence

$$\begin{aligned} E(X_1|S_n) &= E(X_2|S_n) = \cdots = E(X_n|S_n) \qquad a.s. \\ &= n^{-1}E(X_1 + \cdots + X_n|S_n) = n^{-1}E(S_n|S_n) \\ &= n^{-1}S_n \qquad a.s.\,. \end{aligned}$$

5. Show that if $X \in L_1(\Omega, \mathcal{F}, P)$, $Y \in L_1(\Omega, \mathcal{G}, P)$ where $\mathcal{G}$ is a sub-σ field of $\mathcal{F}$, and $E(X1_G) = E(Y1_G)$ for all G in a $\overline{\pi}$-system $\mathcal{G}_0$ generating $\mathcal{G}$, then the same equality holds for all $G \in \mathcal{G}$, and hence $Y = E(X|\mathcal{G})$.
   Hint: use the $\pi - \lambda$ theorem, PfS, Proposition 1.1.5, page 9.

   Solution: As suggested by the hint, one way to do this is via the $\pi - \lambda-$theorem: Let the $\overline{\pi}-$system be denoted by $\mathcal{G}_0$. Let $\mathcal{H} \equiv \{G \in \mathcal{F} : E(X1_G) = E(Y1_G)\}$. Then $\mathcal{G}_0 \subset \mathcal{H}$. Suppose that $A, B \in \mathcal{H}$ with $A \supset B$. Then

$$\begin{aligned} E(X1_{A\setminus B}) &= E(X(1_A - 1_B)) = E(X1_A) - E(X1_B) \\ &= E(Y1_A) - E(Y1_B) = E(Y(1_A - 1_B)) = E(Y1_{A\setminus B}) \end{aligned}$$

   and hence $A \setminus B \in \mathcal{H}$. Now suppose that $\{A_n\} \subset \mathcal{H}$ with $A_n \nearrow A$. Then $E(X1_{A_n}) = E(Y1_{A_n})$ for each n, and we have both $X1_{A_n} \to X1_A$, and $Y1_{A_n} \to Y1_A$, with $|X1_{A_n}| \le |X|$ and $|Y1_{A_n}| \le |Y|$ where $X, Y \in L_1$. Hence, by the above equality for each n and the dominated convergence theorem twice,

$$E(X1_A) = \lim_n E(X1_{A_n}) = \lim_n E(Y1_{A_n}) = E(Y1_A)\,.$$

   Therefore $A \in \mathcal{H}$. Thus $\mathcal{H}$ is a $\lambda-$system. So by the $\pi - \lambda-$ theorem, $\sigma[\mathcal{G}_0] = \mathcal{G} \subset \mathcal{H}$; i.e. the equality holds for all $G \in \mathcal{G}$.

6. **Extra bonus problem: not required**. Let T be the triangle in $\mathbb{R}^2$ where $0 \le y \le x \le 1$, so T has vertices $(0,0)$, $(1,0)$ and $(1,1)$. Let P be the uniform distribution on T, having density with respect to planar Lebesgue measure equal to 2 on T and 0 elsewhere. Let (X,Y) have distribution P. Let $\mathcal{A}$ be the smallest σ algebra for which X is measurable. Show that $E(Y|\mathcal{A}) = X/2$ a.s.

   Solution: Suppose that $A \in \mathcal{A} = \mathcal{F}(X)$. Then we want to show that

$$E\{1_A Y\} = E\{1_A (X/2)\}\,.$$

   But for $A \in \mathcal{A}$ there is a set $B \in \mathcal{B}$ such that $X^{-1}(B) = A$ and hence $1_A = 1_B(X)$. Thus we have, via Fubini's theorem

$$\begin{aligned} E\{1_A Y\} &= E\{1_B(X)Y\} = \int_{B\cap[0,1]} \left(\int_{[0,x]} (2y dy \right) dx \\ &= \int_{B\cap[0,1]} x^2 dx = \int_{B\cap[0,1]} (x/2)(2x) dx = E\{1_B(X)X/2\} \end{aligned}$$

   since the marginal density of X is given by $f_X(x) = 2x1_{[0,1]}(x)$.