Statistics 522, Problem Set 6 Solutions

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1. PfS, Exercise 4.2, page 354: (a) If $\phi''(0$ is finite, then $\sigma^2 < \infty$.
   (b) If $\phi^{(k)}(0)$ is finite, then $E(X^{2k}) < \infty$.

   Solution: (a) First, write

$$\phi'(t) = \lim_{h\to 0} \frac{\phi(t+h) - \phi(t)}{h} = \lim_{h\to 0} \frac{\phi(t) - \phi(t-h)}{h}$$

and

$$\begin{aligned} \phi''(0) &= \lim_{h\to 0} \frac{\phi'(h) - \phi'(0)}{h} = \lim_{h\to 0} \frac{\phi'(0) - \phi'(-h)}{h} \\ &= \lim_{h\to 0} \frac{\phi(h) - 2\phi(0) + \phi(-h)}{h^2} \\ &= \lim_{h\to 0} E\left(\frac{e^{ihX} - 2 + e^{-ihX}}{h^2}\right) \\ &= -2 \lim_{h\to 0} E\left(\frac{1-\cos(hX)}{h^2}\right) . \end{aligned}$$

But

$$\lim_{h\to 0} \frac{1-\cos(hx)}{h^2} = \frac{1}{2}x^2$$

and $(1-\cos(hx))/h^2 \ge 0$ for all x and h. Hence by Fatou's lemma

$$E(X^2) = 2E\left(\lim_{h\to 0}\frac{1-\cos(hX)}{h^2}\right) \le 2\liminf_{h\to 0} E\left(\frac{1-\cos(hX)}{h^2}\right) = -\phi''(0) < \infty\,.$$

Thus $E(X^2) < \infty$ and $Var(X) < \infty$.
(b) The general case follows by induction on k as follows: Suppose that $\phi^{(2k-2)}(0)$ finite implies that $E(X^{2k-2}) < \infty$. Assume that $\phi^{(2k)}(0)$ is finite. Then $\phi^{(2k-2)}(t)$ exists and is continuous in a neighborhood of $t = 0$. By the induction hypothesis, $E(X^{2k-2}) < \infty$. By exercise 4.2 (a),

$$\phi^{(2k-2)}(t) = (-1)^{k-1} E(X^{2k-2} e^{itX})\,.$$

If we define $H(x) \equiv E(X^{2k-2} 1_{[X\le x]})/E(X^{2k-2})$, then $H(x)$ is a d.f. with characteristic function

$$\psi(t) = \int e^{itx} dH(x) = \frac{(-1)^{k-1}\phi^{(2k-2)}(t)}{E(X^{2k-2})}\,.$$

Hence our hypothesis $\phi^{(2k)}(0)$ finite is equivalent to $\psi''(0)$ finite. From (a) it follows that

$$\frac{E(X^{2k})}{E(X^{2k-2})} = E_H(Y^2) \le -\psi''(0) < \infty\,;$$

i.e. $E(X^{2k}) < \infty$.

2. Find independent random variables X, Y, and Z so that Y and Z do not have the same distribution, but $X+Y$ and $X+Z$ do have the same distribution. Hint: let X have the de la Vallee - Poussin distribution with density $f(x) = [1-\cos(x)]/\pi x^2$, let $Y =_d X$, and let Z have the same characteristic function as Y, but extended periodically with period 4; identify the probability distribution of Z explicitly. (This is from Feller, vol. II, pages 505-507.)

   Solution: Let X be a random variable with the de la Vallee - Poussin density $f(x) = (1-\cos(x))/\pi x^2$, and corresponding characteristic function $\phi(t) = (1-|t|)1_{[-1,1]}(t)$. Let Z be a random variable with the charateristic function $\psi(t)$ which is ϕ extended periodically with period 4. (Draw the picture!) Since $\psi(t_0) = 1 = e^{i\theta_0}$ for $t_0 = 4$ and $\theta_0 = 0$ (and for $t_0 = 4k$, $k \in \mathbb{Z}$), it follows from problem 4 below that this characteristic function corresponds to a discrete random variable Z with $P(Z = \pi k/2$ for some $k \in \mathbb{Z}) = 1$. Thus with $p_k \equiv P(Z = \pi k/2)$ the characteristic function ψ is given by

$$\psi(t) = Ee^{itZ} = \sum_{k=-\infty}^{\infty} p_k \exp(it\pi k/2) = \sum_{k=-\infty}^{\infty} p_k \cos(t\pi k/2)$$

   since ψ is real (and hence the p_k's are symmetric about 0). This is a Fourier series, and it is easily seen that the coefficients p_k can be recovered by

$$\begin{aligned} p_k &= P(Z = k\pi/2) = \frac{1}{4}\int_{-2}^{2} \psi(t)\exp(-i\pi kt/2)\,dt \\ &= 2(1-\cos(k\pi/2))/(\pi^2 k^2) \\ &= \begin{cases} \frac{2}{\pi^2 k^2}, & k \in \{1,3,5,\ldots\} \\ \frac{4}{\pi^2 k^2}, & k \in \{2,6,10,\ldots\} \\ 0, & k \in \{4,8,12,\ldots\} \\ 1/4, & k = 0\,. \end{cases} \end{aligned}$$

   This is a discrete version of the de la Valle -Poussin density, and corrects the formula given in Feller (1971) which is missing the factor of k^2 in the denominator. Since $B \equiv \sum_{k=0}^{\infty}(2k+1)^{-2} = \pi^2/8$,

$$\begin{aligned} \sum_{k=-\infty}^{\infty} p_k &= \frac{1}{4} + 2\sum_{k=1}^{\infty} p_k \\ &= \frac{1}{4} + 2\left\{\frac{2}{\pi^2}B + \frac{1}{\pi^2}B\right\} \end{aligned}$$

$$= \frac{1}{4} + 2\left\{\frac{3}{\pi^2}B\right\} = 1\,.$$

See Figure 1.

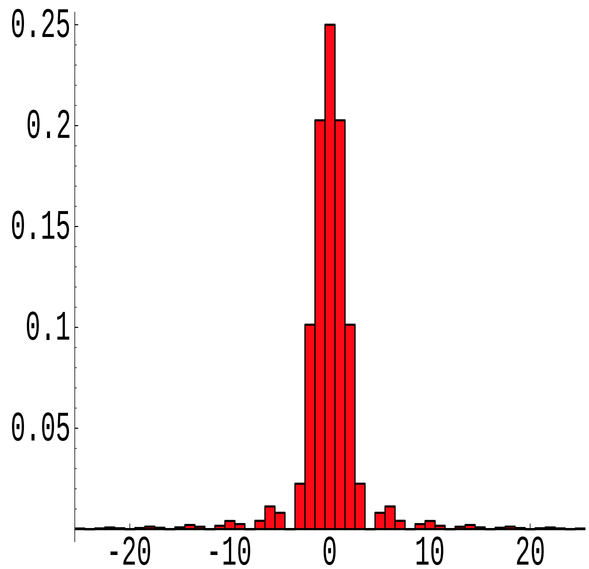

Figure 1: p_k, $k = -25, \ldots, 25$

Then, if $Y =_d X$ so that $\phi_Y = \phi$, we have, by independence

$$\phi_{X+Y}(t) = \phi_X \phi_Y = \phi^2 = \phi\psi = \phi_X\phi_Z = \phi_{X+Z}$$

so that $X + Y =_d X + Z$, but Y and Z have different characteristic functions and hence also different distributions. (This is from Feller, *An Introduction to Probability Theory and Its Applications, Vol. II*, pages 505-507.)

A second solution without characteristic functions: Let $X, \tilde{X} \geq 0$ have the same distribution, and suppose that $Y, \tilde{Y} \geq 0$ have the same distribution where $0 < P(Y = 0) < 1$, $0 < P(X = 0) < 1$, and $X, \tilde{X}, Y, \tilde{Y}$ are independent. Let $Z = -X + \tilde{X} + \tilde{Y}$. Then $P(Z < 0) > 0$ so that Z does not have the same distribution as Y. However $X + Z = X - X + \tilde{X} + \tilde{Y} =_d X + Y$.

3. Suppose that a characteristic function ϕ_X has $|\phi_X(t_0)| = 1$ for some non-zero t_0; i.e. $\phi_X(t_0) = \exp(i\theta_0)$ for some real θ_0. Show that $P(X = (\theta_0 + 2k\pi)/t_0 \text{ for some } k \in \mathbb{Z}) = 1$.
   Hint: Show that $Re(1 - E\exp(it_0X - i\theta_0)) = 0$ and examine the function $1 - \cos(t_0x - \theta_0)$.

Solution: Since $\phi_X(t_0) = \exp(i\theta_0)$ for some $t_0 \neq 0$ and $\theta_0 \in \mathbb{R}$, it follows that

$$E\exp(it_0X - i\theta_0) = 1\,,$$

and hence

$$0 = Re(1 - E\exp(it_0X - i\theta_0)) = E\{1 - \cos(t_0X - \theta_0)\}\,.$$

But the function $f(x) = 1 - \cos(t_0x - \theta_0)$ is non-negative since $|\cos(y)| \le 1$. Thus we have $Ef(X) = 0$ with $f \ge 0$, and hence $f(X) = 0$ a.s. P_X. But this means that $1 - \cos(t_0X - \theta_0) = 0$ a.s., and this occurs only if $t_0X - \theta_0 = k2\pi$ for some integer k a.s. Thus we conclude that

$$P(X = (\theta_0 + 2\pi k)/t_0 \ \text{ for some } k \in \mathbb{Z}) = 1\,.$$

4. PfS, Exercise 1.4 (b). Suppose that $X_{n1}, \ldots, X_{nn}$ are independent Bernoulli(λ_{nk}) random variables for which $\sum_1^n \lambda_{nk} \to \lambda$ and $\sum_1^n \lambda_{nk}^2 \to 0$. Show that

$$M_n \equiv \max_{1\le k\le n} |X_{nk}| \to_d \text{Bernoulli}(1 - e^{-\lambda})\,.$$

Note that theorem 1.2, PfS yields

$$\sum_1^n X_{nk} \to_d Y \sim \text{Poisson}(\lambda)\,.$$

Solution: Let $t \in [0,1)$. then

$$\begin{aligned} P(\max_{1\le k\le n} X_{n,k} \le t) &= P(X_{n,1} \le t, \ldots, X_{n,n} \le t) \\ &= \prod_{k=1}^n P(X_{n,k} \le t) = \prod_{k=1}^n P(X_{n,k} = 0) \\ &= \prod_{k=1}^n (1 - \lambda_{n,k}) \to e^{-\lambda} \end{aligned}$$

by PfS, Lemma 4.3, page 353, since $\sum_{k=1}^n \lambda_{n,k} \to \lambda$, $\delta_n^2 \equiv (\max_k \lambda_{n,k})^2 = \max_{1\le k\le n} \lambda_{n,k}^2 \le \sum_{k=1}^n \lambda_{n,k}^2 \to 0$, and $M_n \equiv \sum_{k=1}^n \lambda_{n,k} \to \lambda > 0$ satisfies $\delta_n M_n \to 0$. For $t \ge 1$, we have $P(\max_{1\le k\le n} X_{n,k} \le t) = 1$ and for $t < 0$, $P(\max_{1\le k\le n} X_{n,k} \le t) = 0$. thus $\max_{1\le k\le n} X_{n,k} \to_d$ Bernoulli($1 - e^{-\lambda}$).