**Statistics 522, Problem Set 2 Solutions**

Wellner; 1/23/2004

1. Suppose that $\tau(X, M)$ is defined as in our proof of the Hartmann-Wintner LIL. Show that for any random variable $X$ with $E(X^2) < \infty$ and any constant $0 < M < \infty$, the following inequality holds:

$$Var(\tau(X, M)) \le Var(X) .$$

Hint: Let $X'$ be an independent copy of $X$. Show that $2Var(\tau(X, M)) = E|\tau(X, M) - \tau(X', M)|^2$ and also that $|\tau(x, M) - \tau(x', M)| \le |x - x'|$ for all real numbers $x$ and $x'$.

**Solution:** For any random variable $Y$ with $E(Y^2) < \infty$ we have

$$2Var(Y) = E(Y - Y')^2 \tag{1}$$

where $Y'$ is an independent copy of $Y$: simply subtract and add $E(Y) = E(Y')$ inside on the right side, expand as $(Y - Y')^2 = (Y - EY - (Y' - EY'))^2 = (Y - EY)^2 - 2(Y - EY)(Y' - EY') + (Y' - EY')^2$, and then use independence of $Y$, $Y'$ to conclude that the expectation of the middle term is zero. Applying this identity to $Y = \tau(X, M)$ yields

$$2Var(\tau(X, M)) = E(\tau(X, M) - \tau(X', M))^2 .$$

By the definition of the function $\tau$ we have

$$\tau(x, M) - \tau(y, M) = \begin{cases} 0, & \text{if } x > M,\ y > M \\ x - M, & \text{if } |x| \le M,\ y > M \\ -2M, & \text{if } x < -M,\ y > M \\ M - y & \text{if } x > M,\ |y| \le M \\ x - y & \text{if } |x| \le M,\ |y| \le M \\ -M - y & \text{if } x < -M,\ |y| \le M \\ 2M & \text{if } x > M,\ y < -M \\ x + M & \text{if } |x| \le M,\ y < -M \\ 0 & \text{if } x < -M,\ y < -M . \end{cases}$$

In every case the absolute value of the difference is less than $|x - y|$. Thus we conclude that $|\tau(x, M) - \tau(y, M)| \le |x - y|$ for all $x, y \in R$. it follows that

$$\begin{aligned} 2Var(\tau(X, M)) &= E(\tau(X, M) - \tau(X', M))^2 \\ &\le E|X - X'|^2 = 2Var(X) \end{aligned}$$

by using (1) again in the last equality.

2. Let $X$ have a Binomial$(n,p)$ distribution, and let $q = 1 - p$. For $0 \le x \le nq$, show that

$$\begin{aligned} P(X \ge np + x) &\le \exp\left(-\frac{x^2}{2npq}\left(q\psi\left(\frac{x}{np}\right) + p\psi\left(\frac{-x}{nq}\right)\right)\right) \\ &\le \exp\left(-\frac{x^2}{2npq}\psi\left(\frac{x(q-p)}{npq}\right)\right) . \end{aligned}$$

Hint: Bound the tail probability by $\exp(-t(np+x) + n\log(q + pe^t))$ for $t \in R^+$, then minimize the expression in the exponential. For $0 < x < nq$ show that the minimum is achieved at $t \equiv \log\left(\frac{1+x/np}{1-x/nq}\right)$. For the second bound, use convexity of $\psi$.

**Solution:** As suggest by the hint, we use to Markov's inequality:

$$\begin{aligned} P(X \ge np + x) &= P(e^{tX} \ge e^{t(np+x)}) \le e^{-t(np+x)} E e^{tX} \\ &= e^{-t(np+x)}(pe^t + q)^n = \exp(-t(np+x) + n\log(pe^t + q)) \\ &\equiv f(t) . \end{aligned}$$

Differentiation of the log of $f$ yields

$$(\log f(t))' = -(np + x) + \frac{npe^t}{pe^t + q} ,$$

and this is zero if

$$\frac{npe^t}{pe^t + q} = np + x$$

or equivalently if

$$\frac{e^t}{pe^t + q} = 1 + x/np ,$$

or equivalently if

$$p + qe^{-t} = (1 + x/np)^{-1} ,$$

or equivalently if

$$t = t_0 \equiv \log\left(\frac{1 + x/np}{1 - x/nq}\right) \tag{2}$$

after a bit of algebra, and noting that we need $1 - x/nq > 0$, or equivalently $x < nq$. Note that for $x \ge nq$ we have $P(X > np + x) \le P(X > np + nq = n) = 0$. Also note that

$$(\log f(t))'' = \frac{np}{(p + qe^{-t})^2}(-1)(-q) > 0 ,$$

so the $t$ in (2) is giving a minimum of $\log f$ and hence also of $f$. Thus we have

$$P(X \ge np + x) \le f(t_0)$$

$$
\begin{aligned}
&= \exp\left(-(np+x)\log\left(\frac{1+x/np}{1-x/nq}\right) + n\log\left(n\left(\frac{1+x/np}{1-x/nq}\right)+q\right)\right) \\
&= \exp\left(-np(1+x/np)\log(1+x/np) + (np+x)\log(1-x/nq)\right. \\
&\qquad\qquad \left. - \, n\log(1-x/nq)\right) \\
&\qquad\qquad \text{since } \; p\frac{1+x/np}{1-x/nq} + q = \frac{1}{1-x/nq}\,, \\
&= \exp\left(-np(1+x/np)\log(1+x/np) - (nq-x)\log(1-x/nq)\right) \\
&= \exp\left(-np\{(1+x/np)\log(1+x/np) - x/np\}\right. \\
&\qquad\qquad \left. - \, nq\{(1-x/nq)\log(1-x/nq) + x/nq\}\right) \\
&= \exp\left(-\frac{x^2}{2npq}\left(q\psi(x/np) + p\psi(-x/nq)\right)\right)
\end{aligned}
$$

since $\psi(x) = 2x^{-2}\{(1+x)\log(1+x) - x\}$. To get the second inequality we use convexity of $\psi$ to conclude that

$$
\begin{aligned}
q\psi(x/np) + p\psi(-x/nq) &\geq \psi\left(q\frac{x}{np} + p\frac{-x}{nq}\right) \\
&= \psi\left(\frac{x}{n}\left(\frac{q}{p} - \frac{p}{q}\right)\right) \\
&= \psi\left(\frac{x}{n}\frac{q^2-p^2}{pq}\right) \\
&= \psi\left(\frac{x}{n}\frac{(q-p)(q+p)}{pq}\right) \\
&= \psi\left(\frac{x(q-p)}{npq}\right).
\end{aligned}
$$

3. Suppose that $Y$ has a Poisson($\lambda$) distribution.
(i) By direct minimization of $\exp(-t(\lambda+x))E\exp(tY)$ over $R^+$ show that

$$
P(Y \geq \lambda + x) \leq \exp\left(-\frac{x^2}{2\lambda}\psi\left(\frac{x}{\lambda}\right)\right).
$$

(ii) Derive the same tail bound by a passage to the limit in the binomial bound from problem 2. (Recall that if $p = p_n \to 0$ and $np_n \to \lambda$, then with $X_n \sim$ Binomial$(n, p_n)$, $X_n \to_d Y \sim$ Poisson($\lambda$).)

**Solution:** (i) As in the previous problem we use Markov's inequality: if $Y \sim$ Poisson($\lambda$) then for $t > 0$

$$
\begin{aligned}
P(Y \geq \lambda + x) &= P(e^{tY} \geq e^{t(\lambda+x)}) \\
&\leq e^{-t(\lambda+x)}Ee^{tY} = e^{-t(\lambda+x)}\exp(\lambda(e^t-1)) \\
&= \exp(-t\lambda - tx + \lambda(e^t - 1)) \equiv f(t)\,.
\end{aligned}
$$

Differentiation of $\log f$ yields

$$(\log f(t))' = -(\lambda + x) + \lambda e^t = 0$$

if $e^t = (\lambda + x)/\lambda = 1 + x/\lambda$. Plugging this into $f$ yields

$$\begin{aligned} P(Y \geq \lambda + x) &\leq \exp(-\lambda(1 + x/\lambda)\log(1 + x/\lambda) + x) \\ &= \exp(-\lambda\{(1 + x/\lambda)\log(1 + x/\lambda) - x/\lambda\}) \\ &= \exp\left(-\frac{x^2}{2\lambda}\psi\left(\frac{x}{\lambda}\right)\right). \end{aligned}$$

(ii) Starting with the second inequality of problem 2 we have

$$P(X \geq np + x) \leq \exp\left(-\frac{x^2}{2npq}\psi\left(\frac{x(q-p)}{npq}\right)\right)$$

for $0 < x < nq$. If we assume that $p = p_n \to 0$ and $np_n \to \lambda > 0$, then, since $X \to_d Y \sim Poisson(\lambda)$, taking limits on both sides of the inequality in the previous display yields

$$P(Y \geq \lambda + x) \leq \exp\left(-\frac{x^2}{2\lambda}\psi\left(\frac{x}{\lambda}\right)\right)$$

as $n \to \infty$ for $0 < x < \infty$.

**Do either problem 4 or problem 5:**

4. *An investment problem.* Suppose that at the beginning of each year you can buy bonds for \$1 that are worth \$a at the end of the year or stocks that are worth a random amount $V \geq 0$. If you always invest a fixed proportion $p$ of your wealth in bonds, then your wealth at the end of year $n+1$ is $W_{n+1} = (ap + (1-p)V_n)W_n$. Suppose that $V, V_1, V_2, \ldots$ are i.i.d. with $EV < \infty$ and $EV^{-2} < \infty$, and that $W_0 = 1$.
(i) Show that $n^{-1}\log W_n \to_{a.s.} c(p)$.
(ii) Show that the limit $c(p)$ is a concave function of $p$. By computing $c'(0)$ and $c'(1)$, give conditions on $V$ that guarantee that the optimal choice of $p$ is in $(0,1)$.
(iii) Suppose that $P(V = 1) = P(V = 4) = 1/2$. Find the optimal $p$ as a function of $a$.

**Solution:** (i) Now

$$W_n = \prod_{j=1}^{n-1}[ap + (1-p)V_j]W_1$$

and hence

$$n^{-1}\log W_n = \frac{1}{n}\sum_{j-1}^{n-1}\log[ap + (1-p)V_j] + n^{-1}\log W_1 \equiv \frac{1}{n}\sum_{j-1}^{n-1}X_j + n^{-1}\log W_1$$

where $X_j \equiv \log(ap + (1-p)V_j)$ are i.i.d. with

$$\begin{aligned} EX_1^+ &= E\left(\{\log(ap+(1-p)V_1)\}1\{ap+(1-p)V_1 \geq 1\}\right) \\ &\leq E\left(\{(ap+(1-p)V_1)\}1\{ap+(1-p)V_1 \geq 1\}\right) \\ &\leq E\{(ap+(1-p)V_1)\} = ap + (1-p)E(V_1) < \infty \end{aligned}$$

since $\log(x) \leq x - 1 \leq x$ and $E(V_1) < \infty$. Also

$$\begin{aligned} EX_1^- &= E\left(\{|\log(ap+(1-p)V_1)|\}1\{ap+(1-p)V_1 < 1\}\right) \\ &\leq E\left([(ap+(1-p)V_1)]^{-2}1\{ap+(1-p)V_1 < 1\}\right) \\ &\leq E\left([(ap+(1-p)V_1)]^{-2}\right) \leq (1-p)^{-2}EV_1^{-2} < \infty \end{aligned}$$

since $|\log(x)| \leq x^{-2}$ for $0 < x \leq 1$ and $EV_1^{-2} < \infty$. Therefore $E|X_1| = EX_1^+ + EX_1^- < \infty$, and by the SLLN

$$\begin{aligned} n^{-1}\log W_n &= \frac{1}{n}\sum_{j-1}^{n-1} X_j + n^{-1}\log W_1 \\ &\to_{a.s.} EX_1 \equiv c(p) = E\log[ap + (1-p)V]\,. \end{aligned}$$

(ii) Set $g(p) \equiv g(p; a, v) \equiv \log(v + (a-v)p)$. Since log is concave, and the composition of a linear function and a concave function is concave, $g$ is a concave function of $p$ for each fixed $a > 0$, $v \geq 0$, and $p \in [0,1]$; for any $\lambda \in [0,1]$, $p_1, p_2 \in [0,1]$,

$$g(\lambda p_1 + \overline{\lambda} p_2) \geq \lambda g(p_1) + \overline{\lambda} g(p_2)\,.$$

Therefore

$$\begin{aligned} c(\lambda p_1 + \overline{\lambda} p_2) &= Eg(\lambda p_1 + \overline{\lambda} p_2; a, V) \\ &\geq E\left\{\lambda g(p_1; a, V) + \overline{\lambda} g(p_2; a, V)\right\} \\ &= \lambda c(p_1) + \overline{\lambda} c(p_2)\,; \end{aligned}$$

i.e. $c(p)$ is concave. Straightforward calculation yields

$$c'(p) = E\left(\frac{a - V}{ap + (1-p)V}\right)\,.$$

If $c'(0) > 0$ and $c'(1) < 0$, then the maximizer of the function $c(p)$ is in $(0,1)$. Taking limits in the above expression for $c'(p)$ yields $c'(1) = E(1 - V/a) < 0$ if $E(V) > a$, and $c'(0) = E(a/V) - 1 > 0$ if $E(1/V) > 1/a$, or if the *harmonic mean of* $V$, $1/E(1/V) < a$. Thus for $1/E(1/V) < a < E(V)$, the maximizer $p_{max} \in (0,1)$.
(iii) In the case that $P(V = 1) = P(V = 4) = 1/2$, $1/E(1/V) = 8/5$ and $EV = 5/2$. Direct computation of $c'(p)$ shows that this is zero for

$$p_{max} \equiv p_{max}(a) = \frac{5a - 8}{2(a-1)(4-a)} \in (0,1)$$

if $8/5 < a < 5/2$. When $a \geq 5/2$, $p_{max}(5/2) = 1$, and the maximum occurs at the boundary. When $a \leq 8/5$, $p_{max} = 0$, and the maximum again occurs at the boundary. (See the attached plots.)

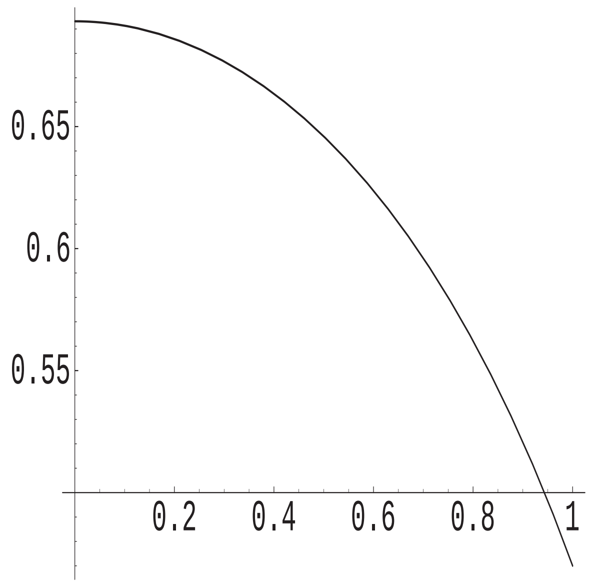


Figure 1: The function $c(p)$ for $a = 8/5$ when $V \sim 3\text{Bernoulli}(1/2) + 1$.

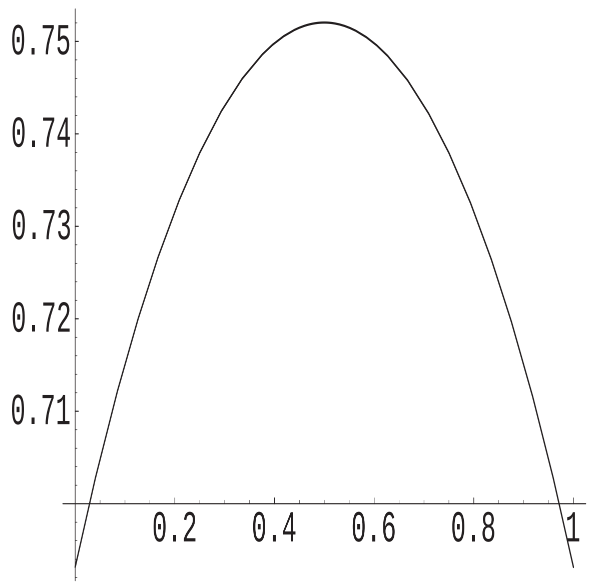


Figure 2: The function $c(p)$ for $a = 2$ when $V \sim 3\text{Bernoulli}(1/2) + 1$.

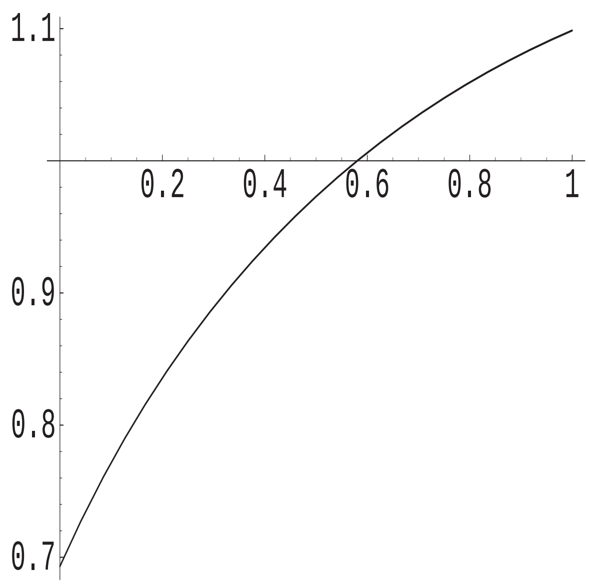


Figure 3: The function $c(p)$ for $a = 3$ when $V \sim 3$ Bernoulli$(1/2) + 1$.

5. (Inversion of Laplace transforms.) Let $P$ be a probability measure on the Borel subsets of $[0,\infty)$, and define its *Laplace transform* by $\varphi(t) = \int_0^\infty e^{-tx} dP(x)$ for $t \in [0,\infty)$. Widder's inversion formula for $P$ from $\varphi$ is:

$$\lim_{n\to\infty} \sum_{k=0}^{[nz]} \frac{(-1)^k}{k!} n^k \varphi^{(k)}(n) = P([0,z)) \tag{3}$$

for $z \in [0,\infty)$ with $P(\{z\}) = 0$. Show that (3) holds via the following steps:
(a) Differentiation of the integral $k$ times shows that

$$\varphi^{(k)}(t) = \int_0^\infty (-x)^k e^{-tx} dP(x)\,.$$

(b) Setting $t = n$, letting $z > 0$, multiplying across by $(-1)^k n^k/k!$, and summing on $k$ yields

$$\sum_{k=0}^{[nz]} \frac{(-1)^k}{k!} n^k \varphi^{(k)}(n) = \int_0^\infty \sum_{k=0}^{[nz]} e^{-nx} \frac{(nx)^k}{k!} dP(x)\,. \tag{4}$$

where $e^{-nx}\frac{(nx)^k}{k!} = P(S_n = k)$ and $S_n = Y_1 + \ldots + Y_n$ where $Y_1, Y_2, \ldots$ are i.i.d. Poisson$(x)$.
(c) Use the weak law of large numbers and (4) to show that (3) holds,

**Solution:** (a) and (b) are self-explanatory and follow immediately. It remains only to show that the limit in (c) holds. Now as noted $e^{-nx}\frac{(nx)^k}{k!} = P(S_n = k)$ and $S_n = Y_1 + \ldots + Y_n$ where $Y_1, Y_2, \ldots$ are i.i.d. Poisson$(x)$. Hence

$$\begin{aligned} \sum_{k=0}^{[nz]} e^{-nx}\frac{(nx)^k}{k!} &= P(S_n \le [nz]) = P(n^{-1}S_n \le n^{-1}[nz]) \\ &= P(n^{-1}S_n - n^{-1}[nz] + z \le z) \\ &= E1\{Y_n \le z\} = Eh_z(Y_n) \end{aligned}$$

where the function $h_z(y) \equiv 1\{y \le z\}$ is bounded and continuous except at the point $y = z$, and where, by the SLLN, $n^{-1}S_n \to_{a.s.} E(Y_1) = x$ and $n^{-1}[nz] \to z$ so that $Y_n \equiv n^{-1}S_n + z - n^{-1}[nz] \to_{a.s.} x$ and hence also $Y_n \to_d x$. By the Helly-Bray Theorem 5.1 it follows that

$$\sum_{k=0}^{[nz]} e^{-nx}\frac{(nx)^k}{k!} = Eh_z(Y_n) \to h_z(x) = 1\{x \le z\}$$

for $x \ne z$. Hence for $z \in [0,\infty)$ with $P(\{z\}) = 0$ we have

$$\sum_{k=0}^{[nz]} \frac{(-1)^k}{k!} n^k \varphi^{(k)}(n) \;=\; \int_0^\infty \sum_{k=0}^{[nz]} e^{-nx}\frac{(nx)^k}{k!} dP(x)$$

$$
\begin{aligned}
&= \int_0^\infty Eh_z(Y_n)dP(x) \\
&\to \int_0^\infty h_z(x)dP(x) \\
&= \int_0^\infty 1\{x \leq z\}dP(x)
\end{aligned}
$$