Statistics 522, Problem Set 1 Solutions

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1. Complete the proof of (7) in Inequality 3.4 on page 212, PfS. i.e. show how the two one-sided arguments combine to yield the inequality with absolute value signs.

   Solution: Let $\tau \equiv \inf\{k \le n : |S_k| \ge \lambda\}$. Thus $[\tau = k] = [|S_1| < \lambda, \ldots |S_{k-1}| < \lambda, |S_k| \ge \lambda]$, $k = 1, \ldots, n$, and $\sum_{k=1}^n [\tau = k] = [\max_{1\le k\le n} |S_k| \ge \lambda]$. Thus, with

   $$a \equiv \min_{1\le k\le n} P(|S_n - S_k| < (1-c)\lambda)\,,$$

   $$\begin{aligned} aP(\max_{1\le k\le n} |S_k| \ge \lambda) &\le \sum_{k=1}^n P(|S_n - S_k| < (1-c)\lambda)P(\tau = k) \\ &= \sum_{k=1}^n P([\tau = k] \cap [|S_n - S_k| < (1-c)\lambda]) \qquad \text{by independence} \\ &= \sum_{k=1}^n P([\tau = k] \cap [|S_k| \ge \lambda] \cap [|S_n - S_k| < (1-c)\lambda]) \\ &\le \sum_{k=1}^n P([\tau = k] \cap [|S_n| \ge c\lambda]) \\ &= P(|S_n| \ge c\lambda)\,. \end{aligned}$$

2. Exercise 10.3.3, page 213, PfS: Suppose that $\epsilon)1, \ldots, \epsilon_n$ are i.i.d. Rademacher random variables. Let $a_1, \ldots, a_n$ be real constants. For $p \ge 1$ it holds that

   $$A_p \left(\sum_1^n a_k^2\right)^{1/2} \le \left(E|\sum_1^n a_k\epsilon_k|^p\right)^{1/p} .$$

   for some constants A_p and B_p. Establish this for $p = 1$ with $A_1 = 1/\sqrt{3}$ and $B_1 = 1$. [Hint: use Littlewood's inequality with r, s, t equal to $4, 2, 1$.]

   Solution: For $p = 1$ it follows from Liapunov's inequality that $E|Y| \le (E|Y|^2)^{1/2}$, so with $Y = \sum_1^n a_k\epsilon_k$

   $$E(Y^2) = \sum_1^n a_k^2\,,$$

and this gives the upper bound with $B_1 = 1$. For the lower bound we use Littlewood's inequality with r, s, t equal to $r, 2, 1$ as suggested by the hint. Thus $m_4 m_1^2 \ge m_2^3$, or

$$E|Y|^2 \le \{E|X|^4\}^{1/3} \{E|X|\}^{2/3} \,,$$

or

$$\frac{\{E|Y|^2\}^{3/2}}{\{E|Y|^4\}^{1/2}} \le E|Y| \,.$$

With $Y = \sum_{i=1}^n a_i \epsilon_i$ we find that $E(Y^2) = \sum_{i=1}^n a_i^2$ and

$$\begin{aligned}
E|X|^4 &= E\{\sum_{j,j',k,k'=1}^n a_j a_{j'} a_k a_{k'} \epsilon_j \epsilon_{j'} \epsilon_k \epsilon_{k'}\} \\
&= \sum_{i=1}^n a_i^4 + \binom{4}{2} \sum_{j<j'} a_j^2 a_{j'}^2 \\
&= \sum_{i=1}^n a_i^4 + 6 \sum_{j<j'} a_j^2 a_{j'}^2 \\
&\le 3(\sum_{i=1}^n a_i^2)^2 \,.
\end{aligned}$$

Hence it follows that

$$E|Y| \ge \frac{\{E|Y|^2\}^{3/2}}{\{E|Y|^4\}^{1/2}} \ge \frac{(\sum a_i^2)^{3/2}}{\sqrt{3} \sum a_i^2} = \frac{1}{\sqrt{3}} (\sum_{i=1}^n a_i^2)^{1/2} \,.$$

We conclude that Khintchine's inequality holds for $p = 1$ with $A_1 = 1/\sqrt{3}$ and $B_1 = 1$. The best possible constants A_p and B_p are known for all p; for $p = 1$ the best possible value of A_p is $1/\sqrt{2}$, and this is due to Szarek (1976), *Studia Math.* **63**, 197-208.

3. Exercise 10.3.4, page 213, PfS. Hint: Use Jensen's inequality. Let $X_1, \ldots, X_n$ be independent with 0 means, and independent of the i.i.d. Rademacher rv's $\epsilon_1, \ldots, \epsilon_n$; thus $P(\epsilon_k = \pm 1) = 1/2$. Let Φ be convex and $\nearrow$ on R. Then

$$E\Phi(|\sum_1^n \epsilon_k X_k|/2) \le E\Phi(|\sum_1^n X_k|) \le E\Phi(2|\sum_1^n \epsilon_k X_k|) \,.$$

Solution: Let $X_1', \ldots, X_n'$ be an independent copy of $X_1, \ldots, X_n$. Then since $E(X_k') = 0$, it follows that

$$E\Phi(|\sum_1^n X_k|) \;=\; E\Phi(|\sum_1^n (X_k - EX_k')|)$$

$$
\begin{aligned}
&\leq E_X \Phi(E_{X'} |\sum_1^n (X_k - X_k')|) \\
&\qquad\qquad \text{by Jensens' inequality since } |\cdot| \text{ is convex} \\
&\leq E_X E_{X'} \Phi(|\sum_1^n (X_k - X_k')|) \qquad \text{since } \Phi \text{ is convex} \\
&= E_{X,X',\epsilon} \Phi(|\sum_1^n \epsilon_k (X_k - X_k')|) \\
&\leq E_{X,X',\epsilon} \Phi(2\{|\sum_1^n \epsilon_k X_k| + |\sum_1^n \epsilon_k X_k'|\}/2) \\
&\qquad \text{since } \Phi \text{ is } \nearrow \\
&\leq \frac{1}{2}\left\{ E\Phi(2\{|\sum_1^n \epsilon_k X_k|) + E\Phi(2\{|\sum_1^n \epsilon_k X_k'|) \right\} \\
&\qquad \text{since } \Phi \text{ is convex} \\
&= E\Phi(2\{|\sum_1^n \epsilon_k X_k|) \\
&\qquad \text{since the two terms have the same expectation.}
\end{aligned}
$$

Thus the second inequality holds (the one on the right). Similarly, to prove the first inequality (the one on the left),

$$
\begin{aligned}
E\Phi(|\sum_1^n \epsilon_k X_k|/2) &= E\Phi(|\sum_1^n \epsilon_k (X_k - EX_k')|/2) \\
&\leq E_{X,\epsilon} \Phi(E_{X'} |\sum_1^n \epsilon_k (X_k - X_k')|/2) \\
&\qquad\qquad \text{by Jensens' inequality since } |\cdot| \text{ is convex} \\
&\leq E_{X,\epsilon} E_{X'} \Phi(|\sum_1^n \epsilon_k (X_k - X_k')|/2) \qquad \text{since } \Phi \text{ is convex} \\
&= E_{X,X'} \Phi(|\sum_1^n (X_k - X_k')|/2) \\
&\leq E_{X,X'} \Phi(\{|\sum_1^n X_k| + |\sum_1^n X_k'|\}/2) \\
&\qquad \text{since } \Phi \text{ is } \nearrow \\
&\leq \frac{1}{2}\left\{ E\Phi(\{|\sum_1^n X_k|) + E\Phi(\{|\sum_1^n X_k'|) \right\} \\
&\qquad \text{since } \Phi \text{ is convex}
\end{aligned}
$$

$$= \quad E\Phi(\{|\sum_1^n \epsilon_k X_k|)$$

since the two terms have the same expectation.

4. (A weak law of large numbers under the assumption of uncorrelated summands.) Suppose that $X_1, X_2, \ldots$ are uncorrelated and $E(X_j^2) \le M < \infty$ for all $j \ge 1$. Show that $\overline{X} - E(\overline{X}_n) = (S_n - ES_n)/n \to_p 0$ and $\overline{X}_n - E(\overline{X}_n) \to_2 0$ as $n \to \infty$.

   Solution: Since the X_n's are uncorrelated,

$$Var(S_n) = \sum_{j=1}^n Var(X_j) + \sum_{i \ne j} Cov(X_i, X_j) = \sum_{j=1}^n Var(X_j) \le \sum_{j=1}^n E(X_j^2) \le nM$$

   and hence $Var(\overline{X}_n) = n^{-2} Var(S_n) \le M/n$. Therefore, by Chebychev's inequality

$$P(|\overline{X}_n - E(\overline{X}_n)| \ge \epsilon) \le \epsilon^{-2} Var(\overline{X}_n) \le \epsilon^{-2} \frac{M}{n} \to 0$$

   for every $\epsilon > 0$; i.e. $\overline{X} - E(\overline{X}_n) \to_p 0$. Since $E[\overline{X}_n - E(\overline{X}_n)]^2 = Var(\overline{X}_n) \le M/n \to 0$, we also have $\overline{X} - E(\overline{X}_n) \to_2 0$.

5. (L_1-convergence in the SLLN.) Suppose that $X_1, \ldots, X_n, \ldots$ are i.i.d with $E|X_1| < \infty$. Show that $\overline{X}_n \to_1 \mu \equiv E(X_1)$; i.e. $E|\overline{X}_n - \mu| \to 0$ as $n \to \infty$. [Hint: show that $|\overline{X}_n| \le Y_n$ where Y_n is uniformly integrable, and that this implies the uniform integrability of $\overline{X}_n$.

   Solution: Note that

$$|\overline{X}_n| \le n^{-1} \sum_{i=1}^n |X_i| \equiv Y_n$$

   where Y_n is uniformly integrable by Vitali's theorem: $E(Y_n) = E|X_1|$ for every n and $Y_n \to_{a.s.} E|X_1| \equiv Y_0$ by the strong law of large numbers and $E(Y_n) \to E(Y_0) = E|X_1|$. It follows that $\{\overline{X}_n\}$ is uniformly integrable, and by Vitali's theorem again $E|\overline{X}_n - \mu| \to 0$ as $n \to \infty$; i.e. $\overline{X}_n \to_1 \mu \equiv E(X_1)$.