Statistics 522, Problem Set 9

Wellner; 3/3/2004

Reading: Shorack, PfS, Chapter 8, pages 158 - 178;
Shorack, PfS, Chapter 10, sections 10.10 and 10.11;
Shorack, PfS, Chapter 18, sections 18.1 - 18.7, pages 467 - 500

Due: Wednesday, March 10, 2004.

1. Exercise 18.3.5, PfS page 477. Let $\{X_n, \mathcal{A}_n\}_{n=0}^{\infty}$ be a sub-martingale with $X_n \geq 0$. Let $r > 1$. Then $\{X_n^r\}$ is uniformly integrable if and only if $\{X_n^r\}$ is integrable.

2. Exercise 18.3.7, PfS page 477. Let $r > 1$. Let $\{X_n, \mathcal{A}_n\}_{n=0}^{\infty}$ be a martingale. Then the following are equivalent:
(10) The $|X_n|^r-$process is integrable.
(11) $X_n \rightarrow_r X_\infty$
(12) The X_n's are uniformly integrable (thus $X_n \rightarrow$ (some X_∞) a.s.) and $X_\infty \in L_r$.
(13) The $|X_n|^r$'s are uniformly integrable.
(14) $\{|X_n|^r, \mathcal{A}_n\}_{n=0}^{\overline{\infty}}$ is a submg and $E|X_n|^r \nearrow E|X_\infty|^r$.

3. Let $X_1, X_2, \ldots$ be i.i.d rv's with $P(X = 1) = p$, $P(X = -1) = 1 - p \equiv q$, where $0 < p < 1$ and $p \neq q$. Suppose that a, b are integers with $-a < 0 < b$. Define
$$S_n = X_1 + \ldots + X_n, \qquad T \equiv \inf\{n : S_n = -a, \text{or} \;\; S_n = b\}\,.$$
Let $\mathcal{F}_n \equiv \sigma[X_1, \ldots, X_n]$, $\mathcal{F}_0 = \{\emptyset, \Omega\}$. Prove that $M_n \equiv (q/p)^{S_n}$ and $N_n = S_n - n(p-q)$ define martingales M_n and N_n. How would you use these martingales to deduce the values of $P(S_T = -a)$ and $E(S_T)$? [Hint: see PfS, pages 499-500.]

4. Exercise 18.7.2, PfS page 500. Suppose tht S_μ is Brownian motion with drift: $S_\mu(t) = S(t) + \mu t$ for $t \geq 0$. Let $\tau_{ab} \equiv \tau \equiv \inf\{t \geq 0 : \;\; S_\mu(t) = -a \text{ or } b\}$ where $-a < 0 < b$.
Claim 1: $S_0(t)$, $S_0^2(t) - t$, $S_\mu(t) - \mu t$ are mean 0 martingales, and, with $\theta = -2\mu$,
$$\exp(\theta[S_\mu(t) - \mu t] - \theta^2 t/2) = \exp(-2\mu[S(t) + \mu t])$$
is a mean 1 martingale.
Claim 2: If $\mu = 0$, $P(S(\tau) = -a) = b/(a+b)$ and $E\tau = ab$.
Claim 3: If $\mu \neq 0$, then
$$P(S(\tau) = -a) = \frac{1 - e^{2\mu b}}{1 - e^{2\mu(a+b)}}$$
and
$$E(\tau) = \frac{b}{\mu} - \frac{a+b}{\mu}\frac{1 - e^{2\mu b}}{1 - e^{2\mu(a+b)}}\,.$$

Claim 4: If $\mu < 0$, then $P(\|S_\mu\|_0^\infty \geq b) = \exp(-2|\mu|b)$ for all $b > 0$; i.e. $\|S_\mu\|_0^\infty \sim$ Exponential$(2|\mu|)$.
(Note the analogies with problem 3.)

5. Suppose that $X_1, X_2, \ldots$ are independent random variables on $(\Omega, \mathcal{A})$ and that X_n has density p_n or q_n under P or Q respectively where p_n and q_n are (for simplicity) everywhere positive on $\mathbb{R}$. Let $\mathcal{F} = \sigma[X_1, X_2, \ldots]$ and $\mathcal{F}_n = \sigma[X_1, \ldots, X_n]$ for $n \geq 1$. Let $Y_n \equiv q_n(X_n)/p_n(X_n)$.
(a) Show that

$$M_n \equiv \left.\frac{dQ}{dP}\right|_{\mathcal{F}_n} = Y_1 \cdots Y_n$$

where the Y_n's are independent and have mean 1 under P; Hence the likelihood ratio martingale of Example 1.14 is the Kakutani product martingale of Example 1.15.
(b) Show that Q is absolutely continuous relative to P on $\mathcal{F}$ if and only if the martingale $\{M_n, \mathcal{F}_n\}$ is uniformly integrable.
(c) Conclude from Kakutani's theorem (PfS Example 4.4, pages 482-483) that $Q \ll P$ on $\mathcal{F}$ if and only if

$$\prod_{n=1}^{\infty} E(Y_n^{1/2}) = \prod_{n=1}^{\infty} \int_{\mathbb{R}} \sqrt{p_n(x) q_n(x)}\, dx > 0\,.$$

(d) Construct two examples of sequences p_n and q_n, one in which the condition in (c) holds and one in which it fails. What is the statistical meaning when it holds and when it fails?