Statistics 522, Problem Set 8

Wellner; 2/25/2004

Reading: Shorack, PfS, Chapter 8, pages 158 - 178;
Shorack, PfS, Chapter 10, sections 10.10 and 10.11;
Shorack, PfS, Chapter 18, sections 18.1 - 18.7, pages 467 - 500

Due: Wednesday, March 3, 2004.

1. PfS, Exercise 18.1.6, page 471.

2. PfS, Exercise 18.3.3, page 477.

3. Suppose that S and T are stopping times relative to the filtration $\{\mathcal{F}_n\}$. Show that $S \wedge T$, $S \vee T$, and $S+T$ are also stopping times.

4. Polyá's urn: At time 0, an urn contains 1 black ball and 1 white ball. At each time $1, 2, 3, \ldots$, a ball is chosen at random from the urn, and is replaced together with a new ball of the same color. Just after time n, there are therefore $n+2$ balls in the urn, of which $B_n + 1$ are black, where B_n is the number of black balls chosen by time n. Let $M_n = (B_n+1)/(n+2)$, the proportion of black balls in the urn just after time n. Prove that (relative to a natural filtration which you should specify) M_n is a martingale. Prove that $P(B_n = k) = 1/(n+1)$ for $0 \le k \le n$. What is the distribution of $\Theta \equiv \lim_n M_n$? Prove that for $0 < \theta < 1$,

$$N_n^\theta \equiv \frac{(n+1)!}{B_n!(n-B_n)!}\theta^{B_n}(1-\theta)^{n-B_n}$$

defines a martingale N_n^θ.

5. Let $\xi_1, \xi_2, \ldots, \xi_n$ be i.i.d. Uniform$(0,1)$ random variables, and let $\mathbb{G}_n(t) = n^{-1}\sum_1^n 1_{[0,t]}(\xi_i)$ for $0 \le t \le 1$; $\mathbb{G}_n$ is the empirical distribution function of the $\xi's$. Fix n and let $\mathcal{F}_n(t) \equiv \sigma\{\mathbb{G}_n(s) : s \le t\}$ for $0 \le t \le 1$. Consider the processes $Y_n(t) \equiv (\mathbb{G}_n(t) - t)/(1-t)$ and $Z_n(t) \equiv (1-\mathbb{G}_n(t))/(1-t)$. Show that $(Y_n(t), \mathcal{F}_n(t) : 0 \le t < 1)$ and $(Z_n(t), \mathcal{F}_n(t) : 0 \le t < 1)$ are martingales.