**Statistics 522, Problem Set 7**

Wellner; 2/18/2004

**Reading:** Shorack, PfS, Chapter 8, pages 158 - 178;
Shorack, PfS, Chapter 10, sections 10.10 and 10.11;
Shorack, PfS, Chapter 18, sections 18.1 - 18.7, pages 467 - 500

**Due:** Wednesday, February 25, 2004.

1. PfS, page 161, Exercise 4.2.

2. PfS, page 161, Exercise 4.3.

3. Suppose that $X, Y \in L_1(\Omega, \mathcal{F}, P)$ and that $E(Y|X) = X$ a.s. and $E(X|Y) = Y$ a.s.. Prove that $P(X = Y) = 1$.

4. Let $X_1, \ldots, X_n$ be i.i.d. real random variables with $E|X_1| < \infty$ and let $S_n \equiv X_1 + \cdots + X_n$. Let $\mathcal{T}_n$ be the smallest $\sigma-$algebra for which $S_k$ are measurable for all $k \geq n$; $\mathcal{T}_n \equiv \sigma\{S_k, k \geq n\}$. Show that $E(X_j|\mathcal{T}_n) = n^{-1}S_n$ a.s. for $j = 1, \ldots, n$.

5. Show that if $X \in L_1(\Omega, \mathcal{F}, P)$, $Y \in L_1(\Omega, \mathcal{G}, P)$ where $\mathcal{G}$ is a sub-$\sigma$ field of $\mathcal{F}$, and $E(X1_G) = E(Y1_G)$ for all $G$ in a $\overline{\pi}$-system $\mathcal{G}_0$ generating $\mathcal{G}$, then the same equality holds for all $G \in \mathcal{G}$, and hence $Y = E(X|\mathcal{G})$.
Hint: use the $\pi - \lambda$ theorem, PfS, Proposition 1.1.5, page 9.

6. **Extra bonus problem: not required**. Let $T$ be the triangle in $\mathbb{R}^2$ where $0 \leq y \leq x \leq 1$, so $T$ has vertices $(0,0)$, $(1,0)$ and $(1,1)$. Let $P$ be the uniform distribution on $T$, having density with respect to planar Lebesgue measure equal to 2 on $T$ and 0 elsewhere. Let $(X,Y)$ have distribution $P$. Let $\mathcal{A}$ be the smallest $\sigma$ algebra for which $X$ is measurable. Show that $E(Y|\mathcal{A}) = X/2$ a.s.