Statistics 522, Problem Set 5

Wellner; 2/4/2004

Reading: Shorack, PfS, Chapter 13, pages 341 - 357;
Shorack, PfS, Chapter 14, pages 365 - 376
Chapter 11, sections 11.3- 11.5.

Due: Wednesday, February 11, 2004.

1. Let $\{X_{n,i}\}$ be a triangular array of random variables, independent with each row and satisfying: (i) $\sum_{i=1}^n P(|X_{n,i}| > \epsilon) \to 0$ for each $\epsilon > 0$,
(ii) $\sum_{i=1}^n Var(X_{n,i} 1_{[|X_{n,i}| \le \epsilon]}) \to 1$ for each $\epsilon > 0$.
Show that $\sum_{i=1}^n X_{n,i} - A_n \to_d Z \sim N(0,1)$ where $A_n \equiv \sum_1^n E(X_{n,i} 1_{[|X_{n,i}| \le 1]})$. Hint: Consider truncated variables $\eta_{n,i} \equiv X_{n,i} 1_{[|X_{n,i}| \le \epsilon_n]}$ and $\xi_{n,i} \equiv \eta_{n,i} - E(\eta_{n,i})$ for an some appropriate sequence ϵ_n.

2. (Liapunov's $2+\delta$ CLT). Suppose that $\{X_{n,i}\}$ is a triangular array of row-wise independent random variables satisfying:
(i) $E(X_{n,i}) = 0$ for $i = 1, \ldots, n$;
(ii) $Var(X_{n,i}) \equiv \sigma_{n,i}^2 < \infty$ for $i = 1, \ldots, n$;
(iii) $\sum_1^n E|X_{n,i}|^{2+\delta}/\sigma_n^{2+\delta} \to 0$ for some $\delta > 0$ where $\sigma_n^2 \equiv \sigma_{n,1}^2 + \cdots + \sigma_{n,n}^2$. Show that $S_n/\sigma_n \to_d Z \sim N(0,1)$. [The classical version of this is with $\delta = 1$.]

3. Construct an example with i.i.d. random variables $X_1, X_2, \ldots$ for which the Lindeberg condition holds but for which Liapunov's $2+\delta$ condition fails for each $0 < \delta \le 1$.

4. Suppose that U_n is a sequence of independent random variables with $P(U_n = \pm cn) = 1/(2n^2)$, $P(U_n = 0) = 1 - 1/n^2$ for some $c > 0$. Let $\{Y_n\}_{n=1}^\infty$ be independent random variables, independent of the U_n, with mean 0 and variance 1 so that $\sqrt{n}\overline{Y}_n \to_d Z \sim N(0,1)$. Consider the independent random variables $X_n = Y_n + U_n$.
(i) Show that with $S_n = X_1 + \cdots X_n$ and $\sigma_n^2 = \sum_1^n Var(X_i)$ $S_n/\sigma_n \to_d aZ \sim N(0, a^2)$ with $a^2 = 1/(1+c^2)$.
(ii) Show that the Lindeberg condition fails.

5. Consider the triangular array of row-wise independent random variables $\{X_{n,i}\}$ with $X_{n,1} \sim N(0, pn)$for some $p \in (0,1)$, $X_{n,j} = 0$ for $2 \le j \le \lfloor pn \rfloor$, and $X_{n,j} \sim N(0,1)$ for $pn < j \le n$. Show that $S_n \sigma_n \to_d Z \sim N(0,1)$ while Lindeberg's condition fails and $\max_{1 \le k \le n} \sigma_{n,k}^2/\sigma_n^2 \to p > 0$.