Statistics 522, Problem Set 4

Wellner; 1/28/2004

Reading: Shorack, PfS, Chapter 11, section 8, pages 292 - 294;
Shorack, PfS, Chapter 13, page 341 - 357;
Chapter 11, sections 11.1- 11.3, pages 1-24.

Due: Wednesday, February 4, 2004.

1. Exercise 7.5, PfS page 290.

2. Suppose that P_n and P are probability measures on $\mathbb{Z}$, the integers, and that $P_n \to_d P$. Show that $d_{TV}(P_n, P) \to 0$ where

$$d_{TV}(P_n, P) = \sup_{B \in \mathcal{B}} |P_n(B) - P(B)| .$$

Hint: You may use the fact that

$$d_{TV}(P_n, P) = \frac{1}{2} \int |p_n - p| d\mu$$

where $p_n = dP_n/d\mu$, $p = dP/d\mu$, and μ is any measure dominating P_n and P; recall Scheffé's theorem, Exercise 5.7, PfS, page 60.

3. Let

$$p_n(x) = 2 \sum_{k=1}^{2^{n-1}} 1_{[(2k-1)/2^n \le x < 2k/2^n]}$$

(draw a picture of p_n!), and let P_n be the probability measures on $\mathbb{R}$ having densities p_n with respect to Lebesgue measure λ.
(a) Show that $P_n \to_d P$.
(b) Show that $\int f dP_n \to \int f dP$ for all bounded measurable functions f on $[0, 1]$. (Not just the continuous ones.)
(c) Find the total variation distance between P_n and $P = \lambda$ on $[0, 1]$.

4. For the following sequences of probability measures on $\mathbb{R}$ have densities p_n with respect to Lebesgue measure, which are uniformly tight? Explain why or why not.
(a) $p_n(x) = n^{-1} 1_{[0,n]}(x)$.
(b) $p_n(x) = n e^{-nx} 1_{[0,\infty)}(x)$.
(c) $p_n(x) = n^{-1} e^{-x/n} 1_{[0,\infty)}(x)$.
(d) $p_n(x) = \sigma_n^{-1} \phi((x - \mu_n)/\sigma_n)$ with $\mu_n = 3 \cdot (-1)^n$ and $\sigma_n^2 = \exp(-\sqrt{n})(2 + \cos(\pi n))$.
(e) $p_n(x) = \sigma_n^{-1} \phi((x - \mu_n)/\sigma_n)$ with $\mu_n = 3 \cdot (-1)^n$ and $\sigma_n^2 = n^{1/3}$.

5. In the previous exercise, identify all the limiting probability distributions or sub-probability distributions (say in terms of the limiting distribution functions or sub-distribution functions) in each case.