Statistics 522, Problem Set 3

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Reading: Shorack, PfS, Chapter 11, section 7, pages 288 - 294;
Chapter 11, sections 11.1- 11.3, pages 1-12.

Due: Wednesday, January 28, 2004.

1. For independent random variables $X_1, \ldots, X_n$, show that

$$P(\max_{1 \le i \le n} |X_i| > x) \ge \frac{\sum_1^n P(|X_i| > x)}{1 + \sum_1^n P(|X_i| > x)}, .$$

In particular, if the left side is less than 1/2, then

$$2P(\max_{1 \le i \le n} |X_i| > x) \ge \sum_{i=1}^n P(|X_i| > x) .$$

2. For $r > 0$, suppose that $X_1, \ldots, X_n$ is a sequence of positive independent random varialbes with $E|X_i|^r < \infty$ for each i. Let $t_0 \equiv \inf\{t > 0 : \sum_1^n P(X_i > t) \le \lambda\}$. Then

$$E \max_{1 \le i \le n} |X_i|^r \begin{cases} \le t_0^r + \sum_{i=1}^n \int_{t_0}^\infty P(X_i > t) d(t^r) \\ \ge \frac{\lambda}{1+\lambda} t_0^r + \frac{1}{1+\lambda} \sum_{i=1}^n \int_{t_0}^\infty P(X_i > t) d(t^r) . \end{cases}$$

Hint: Use problem 1.

3. Let $\epsilon_1, \ldots, \epsilon_n$ be independent Rademacher random variables; i.e. $P(\epsilon_i = \pm 1) = 1/2$. Suppose that $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$. (a) Show that

$$P(|\sum_{i=1}^n a_i \epsilon_i| > x) \le 2 \exp\left(- \frac{x^2}{2\|a\|^2} \right)$$

for all $x > 0$; here $\|a\|^2 = \sum_1^n a_i^2$. This is known as *Hoeffding's inequality*. Hint: Use the fact that $(e^y + e^{-y})/2 \le \exp(y^2/2)$ by writing out the series expansions of both sides.

(b) Use the exponential bound in (a) to show that

$$E|\sum_{i=1}^n a_i \epsilon_i|^4 \le 16 \|a\|^4 .$$

This is a somewhat cruder version of the inequality we established in the course of proving the lower bound part of Khinchine's inequality: in that proof we established

$$E|\sum_{i=1}^n a_i \epsilon_i|^4 \le 3 \|a\|^4$$

by direct calculation.
(c) Use the exponential bound in (a) to show that with $Y \equiv \sum_1^n a_i \epsilon_i$ we have

$$E \exp(tY^2) \le 1 + 2 \frac{t}{\left(\frac{1}{2\|a\|^2} - t \right)}$$

for $t < 1/(2\|a\|^2)$.

4. Exercise 11.7.3, PfS, page 289.