Statistics 522, Problem Set 2

Wellner; 1/14/2004

Reading: Shorack, PfS, Chapter 10, section 11, pages 250-251;
Shorack, PfS, Chapter 11, section 7, pages 288 - 294

Due: Wednesday, January 21, 2004.

1. Suppose that $\tau(X, M)$ is defined as in our proof of the Hartmann-Wintner LIL. Show that for any random variable X with $E(X^2) < \infty$ and any constant $0 < M < \infty$, the following inequality holds:

$$Var(\tau(X, M)) \leq Var(X) .$$

Hint: Let X' be an independent copy of X. Show that $2Var(\tau(X, M)) = E|\tau(X, M) - \tau(X', M)|^2$ and also that $|\tau(x, M) - \tau(x', M)| \leq |x - x'|$ for all real numbers x and x'.

2. Let X have a Binomial(n, p) distribution, and let $q = 1 - p$. For $0 \leq x \leq nq$, show that

$$\begin{aligned} P(X \geq np + x) &\leq \exp\left(-\frac{x^2}{2npq}\left(q\psi\left(\frac{x}{np}\right) + p\psi\left(\frac{-x}{nq}\right)\right)\right) \\ &\leq \exp\left(-\frac{x^2}{2npq}\psi\left(\frac{x(q-p)}{npq}\right)\right) . \end{aligned}$$

Hint: Bound the tail probability by $\exp(-t(np + x) + n\log(q + pe^t))$ for $t \in R^+$, then minimize the expression in the exponential. For $0 < x < nq$ show that the minimum is achieved at $t \equiv \log\left(\frac{1+x/np}{1+x/nq}\right)$. For the second bound, use convexity of ψ.

3. Suppose that Y has a Poisson(λ) distribution.
(i) By direct minimization of $\exp(-t(\lambda + x))E\exp(tY)$ over R^+ show that

$$P(Y \geq \lambda + x) \leq \exp\left(-\frac{x^2}{2\lambda}\psi\left(\frac{x}{\lambda}\right)\right) .$$

(ii) Derive the same tail bound by a passage to the limit in the binomial bound from problem 2. (Recall that if $p = p_n \to 0$ and $np_n \to \lambda$, then with $X_n \sim$ Binomial(n, p_n), $X_n \to_d Y \sim$ Poisson(λ).)

Do either problem 4 or problem 5:

4. *An investment problem.* Suppose that at the beginning of each year you can buy bonds for \$1 that are worth \$a at the end of the year or stocks that are worth a random amount $V \geq 0$. If you always invest a fixed proportion p of your wealth in bonds, then your wealth at the end of year $n+1$ is $W_{n+1} = (ap + (1-p)V_n)W_n$. Suppose that $V, V_1, V_2, \ldots$ are i.i.d. with $EV < \infty$ and $EV^{-2} < \infty$, and that $W_0 = 1$.
(i) Show that $n^{-1} \log W_n \to_{a.s.} c(p)$.
(ii) Show that the limit $c(p)$ is a concave function of p. By computing $c'(0)$ and $c'(1)$, give conditions on V that guarantee that the optimal choice of p is in $(0, 1)$.
(iii) Suppose that $P(V = 1) = P(V = 4) = 1/2$. Find the optimal p as a function of a.

5. (Inversion of Laplace transforms.) Let P be a probability measure on the Borel subsets of $[0, \infty)$, and define its *Laplace transform* by $\varphi(t) = \int_0^\infty e^{-tx} dP(x)$ for $t \in [0, \infty)$. Widder's inversion formula for P from φ is:

$$\lim_{n\to\infty} \sum_{k=0}^{[nz]} \frac{(-1)^k}{k!} n^k \varphi^{(k)}(n) = P([0, z)) \tag{0.1}$$

for $z \in [0, \infty)$ with $P(\{z\}) = 0$. Show that (0.1) holds via the following steps:
(a) Differentiation of the integral k times shows that

$$\varphi^{(k)}(t) = \int_0^\infty (-x)^k e^{-tx} dP(x)\,.$$

(b) Setting $t = n$, letting $z > 0$, multiplying across by $(-1)^k n^k / k!$, and summing on k yields

$$\sum_{k=0}^{[nz]} \frac{(-1)^k}{k!} n^k \varphi^{(k)}(n) = \int_0^\infty \sum_{k=0}^{[nz]} e^{-nx} \frac{(nx)^k}{k!} dP(x)\,. \tag{0.2}$$

where $e^{-nx}\frac{(nx)^k}{k!} = P(S_n = k)$ and $S_n = Y_1 + \ldots + Y_n$ where $Y_1, Y_2, \ldots$ are i.i.d. Poisson(x).
(c) Use the weak law of large numbers and (0.2) to show that (0.1) holds,