Statistics 522, Midterm Exam

Wellner; 2/13/2004

1. (24 points). **Define** *three* of the following six terms:
   (a) A tight probability measure on a metric space (M, d).
   (b) Convergence in distribution of a sequence of measures $\{P_n\}$ on a metric space to a measure P.
   (c) The class of bounded Lipschitz functions $BL(M)$ on a metric space (M, d)?
   (d) The Lévy distance $\lambda(F, G)$ between two distribution functions F and G on $\mathbb{R}$.
   (e) The Prohorov distance $\rho(P, Q)$ between two probability measures P and Q on a metric space (M, d).
   (f) A standard Brownian motion process on $[0, 1]$.

2. (24 points). Give careful **statements** of *any two* of the following five theorems or results:
   (a) The law of the iterated logarithm for i.i.d. random variables $X_1, X_2, \ldots$ with $E(X_1) = 0$ and $Var(X_1) = \sigma^2$.
   (b) Bennett's inequality for bounded random variables.
   (c) The Lindeberg-Feller central limit theorem for a triangular array of row-wise independent random variables with $E(X_{n,i}) = 0$ and $Var(X_{n,i}) = \sigma_{n,i}^2 < \infty$, $i = 1, \ldots, n$.
   (d) Donsker's theorem for the partial sum process $\mathbb{S}_n$ of i.i.d. mean 0, finite variance random variables.
   (e) The theorem of Prohorov and Le Cam relating tightness to relative compactness.

3. (24 points). Let $\phi(z) = (2\pi)^{-1/2} \exp(-z^2/2)$ be the standard Normal density function and let $\Phi(z) = \int_{-\infty}^{z} \phi(t)dt$ be the standard Normal distribution function.
   (a) Prove the upper half of Mills' ratio inequality: i.e. show that

$$1 - \Phi(z) \le \frac{\phi(z)}{z} \qquad \text{for} \quad z > 0\,.$$

   (b) Explain briefly how this is used to prove a part of the law of the iterated logarithm for sums of i.i.d. $N(0, 1)$ random variables.

4. (30 points). Suppose that $X_n \sim \text{Poisson}(n)$; thus $X_n \overset{d}{=} \sum_1^n Y_i$ where Y_i are independent Poisson(1) random variables.
   (a) Compute

$$E\left\{\left(\frac{X_n - n}{\sqrt{n}}\right)^-\right\}$$

explicitly and thereby show that this expectation equals $n^{n+1}e^{-n}/(\sqrt{n}n!)$.
(Recall that $Y^- = -Y 1_{[Y\le 0]}$.)
   (b) Show that $(X_n - n)/\sqrt{n} \to_d Z \sim N(0,1)$.
   (You may appeal to one of our theorems.)
   (c) Use (b) and a uniform integrability argument to show that $E[(X_n - n)^-/\sqrt{n}] \to E[Z^-]$.
   (d) Compute $E[Z^-]$.
   (e) Combine (a) - (d) to show that $n! \sim \sqrt{2\pi n}(n/e)^n$; i.e. $n!/(\sqrt{2\pi n}(n/e)^n) \to 1$.

5. (32 points). Suppose that $X_1, X_2, \ldots$ are i.i.d. random variables with $E(X_1) = 0$, $Var(X_1) = 1$, and let $S_n = X_1 + \cdots + X_n$ for $n \ge 1$. Let $\mathbb{S}_n$ denote the partial sum process,

$$\mathbb{S}_n(t) = \frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} X_i, \qquad 0 \le t \le 1\,.$$

   (a) Rewrite $n^{-3/2}\sum_{i=1}^n S_i$ as a (continuous) function g of $\mathbb{S}_n$.
   (b) Use the result of (a) and Donsker's theorem to show that $n^{-3/2}\sum_{i=1}^n S_i \to_d$ some Y, and find the distribution of Y.
   (c) Use summation by parts and the fact that the X_i's are i.i.d. to show that $n^{-3/2}\sum_{i=1}^n S_i \overset{d}{=} n^{-3/2}\sum_{j=1}^n jX_j$.
   (d) Use the result of (c) together with the Lindeberg-Feller CLT to show that $n^{-3/2}\sum_{i=1}^n S_i \to_d Y$ where Y has the same distribution as was found in (b)