Hoffmann-Jorgensen Inequalities

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The following inequalities, due to J. Hoffmann-Jorgensen (1974), Show that maximal summand plus in "in probability" control of maximal summand actually gives control of moments of sums of independent random variables.

Proposition 1. (Hoffmann-Jorgensen inequalities, probability form). Suppose that $X_1, \ldots, X_n$ are independent random variables, and let $S_k \equiv X_1 + \ldots + X_k$, $k = 1, \ldots, n$. Then for any $\lambda, \eta > 0$,

$$P(\max_{k \le n} |S_k| > 3\lambda + \eta) \le \{P(\max_{k \le n} |S_k| > \lambda)\}^2 + P(\max_{i \le n} |X_i| > \eta).$$

If the X_i's are independent and symmetric, then

$$P(\max_{k \le n} |S_k| > 3\lambda + \eta) \le \{2P(|S_n| > \lambda)\}^2 + P(\max_{i \le n} |X_i| > \eta)$$

and

$$P(|S_n| > 2\lambda + \eta) \le \{2P(|S_n| > \lambda)\}^2 + P(\max_{i \le n} |X_i| > \eta).$$

Proof. Let $\tau \equiv \inf\{k \le n : |S_k| > \lambda\}$. Then $[\tau = k]$ only depends on $X_1, \ldots, X_k$, and $[\max_{k \le n} |S_k| > \lambda] = \sum_{k=1}^n [\tau = k]$. On $[\tau = k]$, $|S_j| \le \lambda]$ if $j < k$, and, for $j \ge k$,

$$|S_j| = |S_j - S_k + X_k + S_{k-1}| \le \lambda + |X_k| + |S_j - S_k|$$

and hence

$$\max_{1 \le j \le n} |S_j| \le \lambda + \max_{i \le n} |X_i| + \max_{k < j \le n} |S_j - S_k|.$$

Therefore, by independence,

$$P(\tau = k, \max_{k \le n} |S_k| > 3\lambda + \eta) \le P(\tau = k, \max_{i \le n} |X_i| > \eta) + P(\tau = k) P(\max_{k < j \le n} |S_j - S_k| > 2\lambda).$$

But $\max_{k < j \le n} |S_j - S_k| \le 2 \max_{k \le n} |S_k|$; and hence summing over k on both sides yields

$$P(\max_{k \le n} |S_k| > 3\lambda + \eta) \le P(\max_{i \le n} |X_i| > \eta) + \{P(\max_{k \le n} |S_k| > \lambda)^2\}.$$

The second inequality follows from the first by Lévy's inequality 3.3. For the symmetric case, first note that

$$|S_n| \leq |S_{k-1}| + |X_k| + |S_n - S_k|,$$

so that

$$P(\tau = k, |S_n| > 2\lambda + \eta) \leq P(\tau = k, \max_{i \leq n} |X_i| > \eta) + P(\tau = k)P(|S_n - S_k| > \lambda);$$

and hence summing over k then yields

$$P(|S_n| > 2\lambda + \eta) \leq P(\max_{i \leq n} |X_i| > \eta) + P(\max_{k \leq n} |S_k| > \lambda)P(\max_{k \leq n} |S_n - S_k| > \lambda).$$

The third inequality then follows from Lévy's inequality 3.3. □

Proposition 2. (Hoffmann-Jorgensen inequalities, moment form). Suppose that $X_1, \ldots, X_n$ are independent random variables, and let $S_k \equiv X_1 + \ldots + X_k$, $k = 1, \ldots, n$. Suppose that $X_k \in L_r(P)$ for some $r > 0$ and each $k = 1, \ldots, n$. Then

$$E\left(\max_{k \leq n} |S_k|^r\right) \leq 2(4t_0)^r + 2 \cdot 4^4 E\left(\max_{i \leq n} |X_i|^r\right)$$

where $t_0 \equiv \inf\{t > 0 : P(\max_{k \leq n} |S_k| > t) \leq 1/(2 \cdot 4^r)\}$. If the X_i's are symmetric, then

$$E|S_n|^r \leq 2 \cdot 3^r E\left(\max_{i \leq n} |X_i|^r\right) + 2(3t_0)^r$$

where $t_0 \equiv \inf\{t > 0 : P(|S_n| > t) \leq 1/(8 \cdot 3^r)\}$.

Proof. Here is the proof of the second inequality; the proof of the first inequality is similar. Let $u > t_0$. Then, using the third inequality (symmetric case) of Proposition 1,

$$\begin{aligned}
E|S_n|^r &= 3^r \int_0^\infty P(|S_n| > 3t)d(t^r) \\
&= 3^r \left(\int_0^u + \int_u^\infty\right) P(|S_n| > 3t)d(t^r) \\
&\leq (3u)^r + 4 \cdot 3^r \int_u^\infty P(|S_n| > t)^2 d(t^r) \\
&\quad + 3^r \int_u^\infty P(\max_{i \leq n} |X_i| > t)d(t^r) \\
&\leq (3u)^r + 4 \cdot 3^r P(|S_n| > u) \int_u^\infty P(|S_n| > t)d(t^r) + 3^r E\left(\max_{i \leq n} |X_i|^r\right).
\end{aligned}$$

Since $4 \cdot 3^r P(|S_n| > u) \leq 1/2$ by our choice of u,

$$E|S_n|^r \leq 2 \cdot (3u)^r + 2 \cdot 3^r E\left(\max_{i \leq n} |X_i|^r\right) .$$