

The Borel Paradox

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Example 1. Suppose that $(X, Y) \sim \text{Uniform}(0, 1)^2$. Is $E(X|X = Y)$ well-defined? (Here we will take $(\Omega, \mathcal{A}, P) = ((0, 1)^2, \mathcal{B}^2 \cap (0, 1)^2, P)$ where P is Lebesgue measure restricted to $(0, 1)^2$.)

Solution 1: Suppose that we try to define $E(X|X = Y)$ by computing $E(X|\mathcal{D})$ where $\mathcal{D} = \{\emptyset, \Omega, D, D^c\}$ where $D^c \equiv \{(x, y) \in (0, 1)^2 : x = y\}$. Then $E(X|\mathcal{D})(x, y)$ is any \mathcal{D} -measurable function of (x, y) (meaning that it is constant on D and D^c) satisfying

$$E(1_D E(X|\mathcal{D})) = E(1_D X), \quad \text{and} \quad E(1_{D^c} E(X|\mathcal{D})) = E(1_{D^c} X).$$

But $E(1_D X) = E(X) = 1/2$ since $P(D^c) = 0$. Hence it is clear that for any $c \in R$

$$E(X|\mathcal{D})(x, y) = \begin{cases} 1/2 & (x, y) \in D \\ c & (x, y) \in D^c \end{cases}$$

is a version of the conditional expectation $E(X|\mathcal{D})$. In this case the “smoothes” version of the conditional expectation is $E(X|\mathcal{D})(x, y) = E(X) = 1/2$ for all $(x, y) \in (0, 1)^2$, but it can be *anything* on the line D^c .

Solution 2: Suppose that we try to define $E(X|X = Y)$ by computing $E(X|\mathcal{D})$ where $\mathcal{D} = \sigma\{Y - X \equiv Z\}$. This can be done easily via example 8.5.1, page 148, and the result is that the regular conditional distribution of $(X|Y - X = Z)$ is given by $P(X \in A|Z = z) = \int_A g(x|z)dx$ for Borel sets A where

$$g(x|z) = \frac{1_{(0,1)}(x)1_{[-z, 1-z]}(w)}{1 - |z|} 1_{(-1,1)}(z).$$

Thus this (regular) version of the conditional distribution has $(X|Z = 0) \sim \text{Uniform}(0, 1)$, so that $E(X|Z = 0) = 1/2$. But once again, $P(Z = 0) = 0$, so $E(X|Z = 0) = c$, c arbitrary, $E(X|Z = z) = 1/2 - z/2$ for $z \neq 0$ is also a version of the conditional expectation conditional on Z .

Solution 3: Suppose that we try to define $E(X|X = Y)$ by computing $E(X|\mathcal{D})$ where $\mathcal{D} = \sigma\{Y/X \equiv Z\}$. This can be done easily via example

8.5.1, page 148, and the result is that the regular conditional distribution of $(X|Z)$ is given by $P(X \in A|Z = z) = \int_A g(x|z)dx$ for Borel sets A where

$$g(x|z) = \frac{2x}{(1/z) \wedge 1} 1_{(0, (1/z) \wedge 1)}(x) 1_{(0, \infty)}(z).$$

It follows that $g(x|1) = 2x 1_{(0,1)}(x)$, the Beta(2, 1) density, which is a different conditional distribution than obtained in Solution 2 by conditioning on $Y - X$. Note that $E(X|Z = 1) = 2/3$ in this regular conditional distribution, but again $P(Z = 1) = 0$, so $E(X|Z = 1) = c$, c arbitrary, $E(X|Z = z) = (2/3)((1/z) \wedge 1)^3$ is also a version of the conditional expectation.

In both Solutions 2 and 3, the σ -field of Solution 1 has been refined (i.e. enlarged), but in two different ways. It is not difficult to find many other ways.

Example 2. See Problem 33.1, page 462, Billingsley, *Probability and Measure*. The following is from Billingsley:

“Suppose that a random point on the unit sphere S^2 in R^3 is specified by longitude Θ and latitude Φ , but restrict Θ to be in $[0, \pi)$, so that Θ only determines the point up to a complete meridian circle (not a semicircle), and to compensate $\phi \in (-\pi, \pi]$.

(a) Show that for given Θ the conditional distribution of Φ has density $(1/4) \cos \phi$ over $(-\pi, \pi]$. If the point lies on, say, the meridian circle through Greenwich, it is thus not uniformly distributed over that great circle.

(b) Show that for given Φ the conditional distribution of Θ is uniform over $(0, \pi)$. If the point lies on the equator (Φ is either 0 or π), it is thus uniformly distributed over that great circle.

Since the point is uniformly distributed over the spherical surface and great circles are indistinguishable, (a) and (b) stand in apparent contradiction. This shows again the inadmissibility of conditioning with respect to an isolated event of probability 0. The relevant σ -field must not be lost sight of.”