The Haar Construction of Brownian Motion and Brownian Bridge

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The aim of this handout is to construct both Brownian motion and Brownian bridge as continuous Gaussian processes on $[0,1]$, and then extend the definition of Brownian motion to $[0,\infty)$.

Definition 1. *Brownian motion* (or *standard Brownian motion*, or, a *Wiener process* $\mathbb{B}$ is a Gaussian process with continuous sample functions and:
(i) $\mathbb{B}(0) = 0$.
(ii) $E\mathbb{B}(t) = 0$, $0 \le t \le 1$.
(iii) $E\{\mathbb{B}(s)\mathbb{B}(t)\} = s \wedge t$, $0 \le s, t \le 1$.

Definition 2. A *Brownian bridge process* $\mathbb{U}$ is a Gaussian process with continuous sample functions and:
(i) $E\mathbb{U}(t) = 0$, $0 \le t \le 1$.
(ii) $E\{\mathbb{U}(s)\mathbb{U}(t)\} = s \wedge t - st$, $0 \le s, t \le 1$.

Thus for any $t_1, \ldots, t_k \in [0,1]$,

$$(\mathbb{U}(t_1), \ldots, \mathbb{U}(t_k)) \sim N_k(\underline{0}, \Sigma_0)$$

where $\Sigma_0 \equiv (t_i \wedge t_j - t_i t_j)$ and

$$(\mathbb{B}(t_1), \ldots, \mathbb{B}(t_k)) \sim N_k(\underline{0}, \Sigma)$$

where $\Sigma \equiv (t_i \wedge t_j)$.

Theorem 1. Brownian motion $\mathbb{B}$ and Brownian bridge $\mathbb{U}$ exist with sample paths in $C[0,1]$ a.s.

Proof. We first construct a Brownian bridge process $\mathbb{U}$.

$$h_{00}(t) \equiv h(t) \equiv \begin{cases} t & 0 \le t \le 1/2 \\ 1-t & 1/2 \le t \le 1 \\ 0 & \text{elsewhere} \end{cases} .$$

Let

$$h_{nj}(t) \equiv 2^{-n/2} h(2^n t - j)\,, j = 0, \ldots, 2^n - 1\,.$$

For example, $h_{10}(t) = 2^{-1/2}h(2t)$, $h_{11}(t) = 2^{-1/2}h(2t-1)$, while,

$$h_{20}(t) = 2^{-1}h(4t), \qquad h_{21}(t) = 2^{-1}h(4t-1),$$

$$h_{22}(t) = 2^{-1}h(4t-2), \qquad h_{23}(t) = 2^{-1}h(4t-3)\,.$$

Note that $|h_{nj}(t)| \le 2^{-n/2}2^{-1}$.

The functions $\{h_{nj}\}_{j=0,n\geq 0}^{2^n-1}$ are called the *Schauder functions*; they are integrals of the orthonormal (with respect to Lebesgue measure on $[0,1]$) family of functions $\{g_{nj}\}_{j=0,n\geq 0}^{2^n-1}$ called the *Haar functions* defined by

$$g_{00}(t) \equiv g(t) \equiv 2 1_{[0,1/2]}(t) - 1\,,$$

$$g_{nj}(t) \equiv 2^{n/2} g_{00}(2^n t - j), \qquad j = 0, \ldots, 2^n - 1, \quad n \geq 1\,.$$

Thus

$$\int_0^1 g_{nj}^2 d\lambda = 1\,,$$

$$\int_0^1 g_{nj} g_{n'j'} d\lambda = 0\,, \qquad \text{if} \quad n \neq n' \quad \text{or} \quad j' \neq j\,,$$

and

$$h_{nj}(t) = \int_0^t g_{nj}(u) du = \int_0^t g_{nj} d\lambda\,, \qquad 0 \leq t \leq 1\,.$$

Furthermore, the family $\{g_{nj}\}_{j=0,n\geq 0}^{2^n-1} \cup \{g(\cdot/2)\}$ is complete: any $f \in L^2[0,1]$ has an expansion in terms of the g's.

Now let $\{X_{nj}\}_{j=0,n\geq 0}^{2^n-1}$ be independent identically distributed $N(0,1)$ random variables; if we wanted, we could construct all these random variables on the probability space $([0,1], \mathcal{B}_{[0,1]}, \lambda)$. Define

$$\mathbb{V}_n(t,\omega) = \sum_{j=0}^{2^n-1} h_{nj}(t) X_{nj}(\omega)\,,$$

$$\mathbb{U}_m(t,\omega) = \sum_{n=0}^{m} \mathbb{V}_n(t,\omega)\,.$$

For $m > k$,

$$|\mathbb{U}_m(t,\omega) - \mathbb{U}_k(t,\omega)| = |\sum_{n=k+1}^{m} \mathbb{V}_n(t,\omega)| \leq \sum_{n=k+1}^{m} |\mathbb{V}_n(t,\omega)|$$

where

$$|\mathbb{V}_n(t,\omega)| \leq \sum_{j=0}^{2^n-1} |h_{nj}(t)||X_{nj}(\omega)| \leq 2^{-n/2-1} \max_{0\leq j\leq 2^n-1} |X_{nj}(\omega)|$$

since the h_{nj}, $j = 0, \ldots, 2^n - 1$ are $\neq 0$ on disjoint t intervals.

Now $P(X_{nj} \geq z) = 1 - \Phi(z) \leq z^{-1}\phi(z)$ for $z > 0$ (Mills' ratio), so that

$$P(|X_{nj}| \geq 2\sqrt{n}) = 2P(X_{nj} \geq 2\sqrt{n}) \leq \frac{2}{\sqrt{2\pi}} (2\sqrt{n})^{-1} e^{-2n}\,.$$

Hence

$$P(\max_{0\leq j\leq 2^n-1} |X_{nj}| \geq 2\sqrt{n}) \leq 2^n P(|X_{00}| \geq 2\sqrt{n}) \leq \frac{2^n}{\sqrt{2\pi}} n^{-1/2} e^{-2n}\,;$$

since this is a term of a convergent series, by the Borel-Cantelli lemma, $\max_{0\le j\le 2^n-1}|X_{nj}|\ge 2\sqrt{n}$ occurs infinitely often with probability zero: i.e. except on a null set, for all ω there is an $N\equiv N(\omega)$ such that $\max_{0\le j\le 2^n-1}|X_{nj}(\omega)|<2\sqrt{n}$ for all $n\ge N(\omega)$. Hence

$$\sup_{0\le t\le 1}|\mathbb{U}_m(t)-\mathbb{U}_k(t)|\le \sum_{n=k+1}^{m} 2^{-n/2}n^{1/2}\downarrow 0$$

for all $k,m\ge N'\ge N(\omega)$. Thus $\mathbb{U}_m(t,\omega)$ converges uniformly as $m\to\infty$ with probability one to (the necessarily continuous) function

$$\mathbb{U}(t,\omega)\equiv\sum_{n=0}^{\infty}\mathbb{V}_n(t,\omega)\,.$$

Define $\mathbb{U}\equiv 0$ on the exceptional set. Then $\mathbb{U}$ is continuous for *all* ω.

Now $\{\mathbb{U}(t):0\le t\le 1\}$ is clearly a Gaussian process since it is the sum of Gaussian processes. We now show that $\mathbb{U}$ is in fact a Brownian bridge: by formal calculation (it remains only to justify the interchange of summation and expectation),

$$\begin{aligned}
E\{\mathbb{U}(s)\mathbb{U}(t)\} &= E\{\sum_{n=0}^{\infty}\mathbb{V}_n(s)\sum_{m=0}^{\infty}\mathbb{V}_m(t)\}\\
&= \sum_{n=0}^{\infty}E\{\mathbb{V}_n(s)\mathbb{V}_n(t)\}\\
&= \sum_{n=0}^{\infty}E\{\sum_{j=0}^{2^n-1}X_{nj}\int_0^s g_{nj}\sum_{k=0}^{2^n-1}X_{nk}\int_0^t g_{nk}\}\\
&= \sum_{n=0}^{\infty}\sum_{j=0}^{2^n-1}\int_0^s g_{nj}\int_0^t g_{nj}\\
&= \sum_{n=0}^{\infty}\sum_{j=0}^{2^n-1}\int_0^1 1_{[0,s]}g_{nj}\int_0^1 1_{[0,t]}g_{nj}+st-st\\
&= \int_0^1 1_{[0,s]}(u)1_{[0,t]}(u)\,du-st\\
&= s\wedge t-st
\end{aligned}$$

where the next to last equality follows from Parseval's identity. Thus $\mathbb{U}$ is Brownian bridge.

Now let X be one additional $N(0,1)$ rv independent of all the others used in the construction, and define

$$\mathbb{B}(t)\equiv\mathbb{U}(t)+tX=\sum_{n=0}^{\infty}\mathbb{V}_n(t)+tX\,.$$

Then $\mathbb{B}$ is also Gaussian with 0 mean and

$$Cov(\mathbb{B}(s),\mathbb{B}(t)) = Cov(\mathbb{U}(s)+sX,\mathbb{U}(t)+tX)$$

$$
\begin{aligned}
&= Cov(\mathbb{U}(s), \mathbb{U}(t)) + stVar(X) \\
&= s \wedge t - st + st = s \wedge t\,.
\end{aligned}
$$

Thus $\mathbb{B}$ is Brownian motion; since $\mathbb{U}$ has continuous sample paths, so does $\mathbb{B}$. □

Exercise 1. Graph the first few g_{nj}'s and h_{nj}'s.

Exercise 2. Justify the interchange of expectation and summation used in the proof. [Hint: use the Tonelli part of Fubini's theorem.]

Exercise 3. Let $\mathbb{U}$ be a Brownian bridge process. For $0 \le t < \infty$ define a process $\mathbb{B}$ by

$$\mathbb{B}(t) \equiv (1+t)\mathbb{U}(\frac{t}{1+t})\,.$$

Show that $\mathbb{B}$ is a Brownian motion process on $[0, \infty)$.