

The Haar Function Construction of Brownian Motion and Brownian Bridge

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The aim of this handout is to construct both Brownian motion and Brownian bridge as continuous Gaussian processes on $[0, 1]$, and then extend the definition of Brownian motion to $[0, \infty)$.

Definition 1. *Brownian motion* (or *standard Brownian motion*, or, a *Wiener process* \mathbb{B} is a Gaussian process with continuous sample functions and:

- (i) $\mathbb{B}(0) = 0$.
- (ii) $E\mathbb{B}(t) = 0, 0 \leq t \leq 1$.
- (iii) $E\{\mathbb{B}(s)\mathbb{B}(t)\} = s \wedge t, 0 \leq s, t \leq 1$.

Definition 2. A *Brownian bridge process* \mathbb{U} is a Gaussian process with continuous sample functions and:

- (i) $E\mathbb{U}(t) = 0, 0 \leq t \leq 1$.
- (ii) $E\{\mathbb{U}(s)\mathbb{U}(t)\} = s \wedge t - st, 0 \leq s, t \leq 1$.

Thus for any $t_1, \dots, t_k \in [0, 1]$,

$$(\mathbb{U}(t_1), \dots, \mathbb{U}(t_k)) \sim N_k(\mathbf{0}, \Sigma_0)$$

where $\Sigma_0 \equiv (t_i \wedge t_j - t_i t_j)$ and

$$(\mathbb{B}(t_1), \dots, \mathbb{B}(t_k)) \sim N_k(\mathbf{0}, \Sigma)$$

where $\Sigma \equiv (t_i \wedge t_j)$.

Theorem 1. Brownian motion \mathbb{B} and Brownian bridge \mathbb{U} exist with sample paths in $C[0, 1]$ a.s.

Proof. We first construct a Brownian bridge process \mathbb{U} .

$$h_{00}(t) \equiv h(t) \equiv \begin{cases} t & 0 \leq t \leq 1/2 \\ 1-t & 1/2 \leq t \leq 1 \\ 0 & \text{elsewhere} \end{cases} .$$

Let

$$h_{nj}(t) \equiv 2^{-n/2} h(2^n t - j), j = 0, \dots, 2^n - 1.$$

For example, $h_{10}(t) = 2^{-1/2} h(2t)$, $h_{11}(t) = 2^{-1/2} h(2t - 1)$, while,

$$h_{20}(t) = 2^{-1} h(4t), \quad h_{21}(t) = 2^{-1} h(4t - 1),$$

$$h_{22}(t) = 2^{-1} h(4t - 2), \quad h_{23}(t) = 2^{-1} h(4t - 3).$$

Note that $|h_{nj}(t)| \leq 2^{-n/2} 2^{-1}$.

The functions $\{h_{nj}\}_{j=0, n \geq 0}^{2^n-1}$ are called the *Schauder functions*; they are integrals of the orthonormal (with respect to Lebesgue measure on $[0, 1]$) family of functions $\{g_{nj}\}_{j=0, n \geq 0}^{2^n-1}$ called the *Haar functions* defined by

$$g_{00}(t) \equiv g(t) \equiv 21_{[0,1/2]}(t) - 1,$$

$$g_{nj}(t) \equiv 2^{n/2}g_{00}(2^n t - j), \quad j = 0, \dots, 2^n - 1, \quad n \geq 1.$$

Thus

$$\int_0^1 g_{nj}^2 d\lambda = 1,$$

$$\int_0^1 g_{nj}g_{n'j'} d\lambda = 0, \quad \text{if } n \neq n' \text{ or } j' \neq j,$$

and

$$h_{nj}(t) = \int_0^t g_{nj}(u) du = \int_0^t g_{nj} d\lambda, \quad 0 \leq t \leq 1.$$

Furthermore, the family $\{g_{nj}\}_{j=0, n \geq 0}^{2^n-1} \cup \{g(\cdot/2)\}$ is complete: any $f \in L^2[0, 1]$ has an expansion in terms of the g 's.

Now let $\{X_{nj}\}_{j=0, n \geq 0}^{2^n-1}$ be independent identically distributed $N(0, 1)$ random variables; if we wanted, we could construct all these random variables on the probability space $([0, 1], \mathcal{B}_{[0,1]}, \lambda)$. Define

$$\mathbb{V}_n(t, \omega) = \sum_{j=0}^{2^n-1} h_{nj}(t) X_{nj}(\omega),$$

$$\mathbb{U}_m(t, \omega) = \sum_{n=0}^m \mathbb{V}_n(t, \omega).$$

For $m > k$,

$$|\mathbb{U}_m(t, \omega) - \mathbb{U}_k(t, \omega)| = \left| \sum_{n=k+1}^m \mathbb{V}_n(t, \omega) \right| \leq \sum_{n=k+1}^m \left| \mathbb{V}_n(t, \omega) \right|$$

where

$$|\mathbb{V}_n(t, \omega)| \leq \sum_{j=0}^{2^n-1} |h_{nj}(t)| |X_{nj}(\omega)| \leq 2^{-n/2-1} \max_{0 \leq j \leq 2^n-1} |X_{nj}(\omega)|$$

since the h_{nj} , $j = 0, \dots, 2^n - 1$ are $\neq 0$ on disjoint t intervals.

Now $P(X_{nj} \geq z) = 1 - \Phi(z) \leq z^{-1}\phi(z)$ for $z > 0$ (Mills' ratio), so that

$$P(|X_{nj}| \geq 2\sqrt{n}) = 2P(X_{nj} \geq 2\sqrt{n}) \leq \frac{2}{\sqrt{2\pi}} (2\sqrt{n})^{-1} e^{-2n}.$$

Hence

$$P\left(\max_{0 \leq j \leq 2^n-1} |X_{nj}| \geq 2\sqrt{n}\right) \leq 2^n P(|X_{00}| \geq 2\sqrt{n}) \leq \frac{2^n}{\sqrt{2\pi}} n^{-1/2} e^{-2n};$$

since this is a term of a convergent series, by the Borel-Cantelli lemma, $\max_{0 \leq j \leq 2^n - 1} |X_{nj}| \geq 2\sqrt{n}$ occurs infinitely often with probability zero: i.e. except on a null set, for all ω there is an $N \equiv N(\omega)$ such that $\max_{0 \leq j \leq 2^n - 1} |X_{nj}(\omega)| < 2\sqrt{n}$ for all $n \geq N(\omega)$. Hence

$$\sup_{0 \leq t \leq 1} |\mathbb{U}_m(t) - \mathbb{U}_k(t)| \leq \sum_{n=k+1}^m 2^{-n/2} n^{1/2} \downarrow 0$$

for all $k, m \geq N' \geq N(\omega)$. Thus $\mathbb{U}_m(t, \omega)$ converges uniformly as $m \rightarrow \infty$ with probability one to (the necessarily continuous) function

$$\mathbb{U}(t, \omega) \equiv \sum_{n=0}^{\infty} \mathbb{V}_n(t, \omega).$$

Define $\mathbb{U} \equiv 0$ on the exceptional set. Then \mathbb{U} is continuous for *all* ω .

Now $\{\mathbb{U}(t) : 0 \leq t \leq 1\}$ is clearly a Gaussian process since it is the sum of Gaussian processes. We now show that \mathbb{U} is in fact a Brownian bridge: by formal calculation (it remains only to justify the interchange of summation and expectation),

$$\begin{aligned} E\{\mathbb{U}(s)\mathbb{U}(t)\} &= E\left\{\sum_{n=0}^{\infty} \mathbb{V}_n(s) \sum_{m=0}^{\infty} \mathbb{V}_m(t)\right\} \\ &= \sum_{n=0}^{\infty} E\{\mathbb{V}_n(s)\mathbb{V}_n(t)\} \\ &= \sum_{n=0}^{\infty} E\left\{\sum_{j=0}^{2^n-1} X_{nj} \int_0^s g_{nj}(u) du \sum_{k=0}^{2^n-1} X_{nk} \int_0^t g_{nk}(v) dv\right\} \\ &= \sum_{n=0}^{\infty} \sum_{j=0}^{2^n-1} \int_0^s g_{nj}(u) du \int_0^t g_{nj}(v) dv \\ &= \sum_{n=0}^{\infty} \sum_{j=0}^{2^n-1} \int_0^1 1_{[0,s]}(u) g_{nj}(u) du \int_0^1 1_{[0,t]}(v) g_{nj}(v) dv + st - st \\ &= \int_0^1 1_{[0,s]}(u) 1_{[0,t]}(u) du - st \\ &= s \wedge t - st \end{aligned}$$

where the next to last equality follows from Parseval's identity. Thus \mathbb{U} is Brownian bridge.

Now let X be one additional $N(0, 1)$ rv independent of all the others used in the construction, and define

$$\mathbb{B}(t) \equiv \mathbb{U}(t) + tX = \sum_{n=0}^{\infty} \mathbb{V}_n(t) + tX.$$

Then \mathbb{B} is also Gaussian with 0 mean and

$$\begin{aligned} \text{Cov}(\mathbb{B}(s), \mathbb{B}(t)) &= \text{Cov}(\mathbb{U}(s) + sX, \mathbb{U}(t) + tX) \\ &= \text{Cov}(\mathbb{U}(s), \mathbb{U}(t)) + st\text{Var}(X) \\ &= s \wedge t - st + st = s \wedge t. \end{aligned}$$

Thus \mathbb{B} is Brownian motion; since \mathbb{U} has continuous sample paths, so does \mathbb{B} . □

Exercise 1. Graph the first few g_{nj} 's and h_{nj} 's.

Exercise 2. Justify the interchange of expectation and summation used in the proof. [Hint: use the Tonelli part of Fubini's theorem.]

Exercise 3. Let \mathbb{U} be a Brownian bridge process. For $0 \leq t < \infty$ define a process \mathbb{B} by

$$\mathbb{B}(t) \equiv (1+t)\mathbb{U}\left(\frac{t}{1+t}\right).$$

Show that \mathbb{B} is a Brownian motion process on $[0, \infty)$.