

## The Haar Construction of Brownian Motion and Brownian Bridge

Wellner; 4/26/99; 2/27/08

The aim of this handout is to construct both Brownian motion and Brownian bridge as continuous Gaussian processes on  $[0, 1]$ , and that we can then extend the definition of Brownian motion to  $[0, \infty)$ .

**Definition.** *Brownian motion* (or *standard Brownian motion*, or, a *Wiener process*  $\mathbb{B}$  is a Gaussian process with continuous sample functions and:

- (i)  $\mathbb{B}(0) = 0$ .
- (ii)  $E\mathbb{B}(t) = 0, 0 \leq t \leq 1$ .
- (iii)  $E\{\mathbb{B}(s)\mathbb{B}(t)\} = s \wedge t, 0 \leq s, t \leq 1$ .

**Definition.** A *Brownian bridge process*  $\mathbb{U}$  is a Gaussian process with continuous sample functions and:

- (i)  $E\mathbb{U}(t) = 0, 0 \leq t \leq 1$ .
- (ii)  $E\{\mathbb{U}(s)\mathbb{U}(t)\} = s \wedge t - st, 0 \leq s, t \leq 1$ .

Thus for any  $t_1, \dots, t_k \in [0, 1]$ ,

$$(\mathbb{U}(t_1), \dots, \mathbb{U}(t_k)) \sim N_k(\underline{0}, \Sigma_0)$$

where  $\Sigma_0 \equiv (t_i \wedge t_j - t_i t_j)$  and

$$(\mathbb{B}(t_1), \dots, \mathbb{B}(t_k)) \sim N_k(\underline{0}, \Sigma)$$

where  $\Sigma \equiv (t_i \wedge t_j)$ .

**Theorem.** Brownian motion  $\mathbb{B}$  and Brownian bridge  $\mathbb{U}$  exist. with sample paths in  $C[0, 1]$  a.s.

**Proof.** We first construct a Brownian bridge process  $\mathbb{U}$ .

$$h_{00}(t) \equiv h(t) \equiv \begin{cases} t & 0 \leq t \leq 1/2 \\ 1-t & 1/2 \leq t \leq 1 \\ 0 & \text{elsewhere} \end{cases} .$$

Let

$$h_{nj}(t) \equiv 2^{-n/2}h(2^n t - j), j = 0, \dots, 2^n - 1.$$

For example,  $h_{10}(t) = 2^{-1/2}h(2t)$ ,  $h_{11}(t) = 2^{-1/2}h(2t - 1)$ , while,

$$h_{20}(t) = 2^{-1}h(4t), \quad h_{21}(t) = 2^{-1}h(4t - 1),$$

$$h_{22}(t) = 2^{-1}h(4t - 2), \quad h_{23}(t) = 2^{-1}h(4t - 3).$$

Note that  $|h_{nj}(t)| \leq 2^{-n/2}2^{-1}$ .

The functions  $\{h_{nj}\}_{j=0, n \geq 0}^{2^n-1}$  are called the *Schauder functions*; they are integrals of the orthonormal (with respect to Lebesgue measure on  $[0, 1]$ ) family of functions  $\{g_{nj}\}_{j=0, n \geq 0}^{2^n-1}$  called the *Haar functions* defined by

$$g_{00}(t) \equiv g(t) \equiv 21_{[0,1/2]}(t) - 1,$$

$$g_{nj}(t) \equiv 2^{n/2}g_{00}(2^n t - j), \quad j = 0, \dots, 2^n - 1, \quad n \geq 1.$$

Thus

$$\int_0^1 g_{nj}^2 d\lambda = 1,$$

$$\int_0^1 g_{nj} g_{n'j'} d\lambda = 0, \quad \text{if } n \neq n' \text{ or } j' \neq j,$$

and

$$h_{nj}(t) = \int_0^t g_{nj}(u) du = \int_0^t g_{nj} d\lambda, \quad 0 \leq t \leq 1.$$

Furthermore, the family  $\{g_{nj}\}_{j=0, n \geq 0}^{2^n-1} \cup \{g(\cdot/2)\}$  is complete: any  $f \in L^2[0, 1]$  has an expansion in terms of the  $g$ 's.

Now let  $\{X_{nj}\}_{j=0, n \geq 0}^{2^n-1}$  be independent identically distributed  $N(0, 1)$  random variables; if we wanted, we could construct all these random variables on the probability space  $([0, 1], \mathcal{B}_{[0,1]}, \lambda)$ . Define

$$\mathbb{V}_n(t, \omega) = \sum_{j=0}^{2^n-1} h_{nj}(t) X_{nj}(\omega),$$

$$\mathbb{U}_m(t, \omega) = \sum_{n=0}^m \mathbb{V}_n(t, \omega).$$

For  $m > k$ ,

$$|\mathbb{U}_m(t, \omega) - \mathbb{U}_k(t, \omega)| = \left| \sum_{n=k+1}^m \mathbb{V}_n(t, \omega) \right| \leq \sum_{n=k+1}^m |\mathbb{V}_n(t, \omega)|$$

where

$$|\mathbb{V}_n(t, \omega)| \leq \sum_{j=0}^{2^n-1} |h_{nj}(t)| |X_{nj}(\omega)| \leq 2^{-n/2-1} \max_{0 \leq j \leq 2^n-1} |X_{nj}(\omega)|$$

since the  $h_{nj}$ ,  $j = 0, \dots, 2^n - 1$  are  $\neq 0$  on disjoint  $t$  intervals.

Now  $P(X_{nj} \geq z) = 1 - \Phi(z) \leq z^{-1}\phi(z)$  for  $z > 0$  (Mill's ratio), so that

$$P(|X_{nj}| \geq 2\sqrt{n}) = 2P(X_{nj} \geq 2\sqrt{n}) \leq \frac{2}{\sqrt{2\pi}} (2\sqrt{n})^{-1} e^{-2n}.$$

Hence

$$P\left(\max_{0 \leq j \leq 2^n-1} |X_{nj}| \geq 2\sqrt{n}\right) \leq 2^n P(|X_{00}| \geq 2\sqrt{n}) \leq \frac{2^n}{\sqrt{2\pi}} n^{-1/2} e^{-2n};$$

since this is a term of a convergent series, by the Borel-Cantelli lemma,  $\max_{0 \leq j \leq 2^n-1} |X_{nj}| \geq 2\sqrt{n}$  occurs infinitely often with probability zero: i.e. except on a null set, for all  $\omega$  there is an  $N \equiv N(\omega)$  such that  $\max_{0 \leq j \leq 2^n-1} |X_{nj}(\omega)| < 2\sqrt{n}$  for all  $n \geq N(\omega)$ . Hence

$$\sup_{0 \leq t \leq 1} |\mathbb{U}_m(t) - \mathbb{U}_k(t)| \leq \sum_{n=k+1}^m 2^{-n/2} n^{1/2} \downarrow 0$$

for all  $k, m \geq N' \geq N(\omega)$ . Thus  $\mathbb{U}_m(t, \omega)$  converges uniformly as  $m \rightarrow \infty$  with probability one to (the necessarily continuous) function

$$\mathbb{U}(t, \omega) \equiv \sum_{n=0}^{\infty} \mathbb{V}_n(t, \omega).$$

Define  $\mathbb{U} \equiv 0$  on the exceptional set. Then  $\mathbb{U}$  is continuous for *all*  $\omega$ .

Now  $\{\mathbb{U}(t) : 0 \leq t \leq 1\}$  is clearly a Gaussian process since it is the sum of Gaussian processes. We now show that  $\mathbb{U}$  is in fact a Brownian bridge: by

formal calculation (it remains only to justify the interchange of summation and expectation),

$$\begin{aligned}
E\{\mathbb{U}(s)\mathbb{U}(t)\} &= E\left\{\sum_{n=0}^{\infty} \mathbb{V}_n(s) \sum_{m=0}^{\infty} \mathbb{V}_m(t)\right\} \\
&= \sum_{n=0}^{\infty} E\{\mathbb{V}_n(s)\mathbb{V}_n(t)\} \\
&= E\left\{\sum_{j=0}^{2^n-1} X_{nj} \int_0^s g_{nj} \sum_{k=0}^{2^n-1} X_{nk} \int_0^t g_{nk}\right\} \\
&= \sum_{n=0}^{\infty} \sum_{j=0}^{2^n-1} \int_0^s g_{nj} \int_0^t g_{nj} \\
&= \sum_{n=0}^{\infty} \sum_{j=0}^{2^n-1} \int_0^1 1_{[0,s]} g_{nj} \int_0^1 1_{[0,t]} g_{nj} + st - st \\
&= \int_0^1 1_{[0,s]}(u) 1_{[0,t]}(u) du - st \\
&= s \wedge t - st
\end{aligned}$$

where the next to last equality follows from Parseval's identity. Thus  $\mathbb{U}$  is Brownian bridge.

Now let  $X$  be one additional  $N(0, 1)$  rv independent of all the others used in the construction, and define

$$\mathbb{B}(t) \equiv \mathbb{U}(t) + tX = \sum_{n=0}^{\infty} \mathbb{V}_n(t) + tX.$$

Then  $\mathbb{B}$  is also Gaussian with 0 mean and

$$\begin{aligned}
Cov(\mathbb{B}(s), \mathbb{B}(t)) &= Cov(\mathbb{U}(s) + sX, \mathbb{U}(t) + tX) \\
&= Cov(\mathbb{U}(s), \mathbb{U}(t)) + stVar(X) \\
&= s \wedge t - st + st = s \wedge t.
\end{aligned}$$

Thus  $\mathbb{B}$  is Brownian motion; since  $\mathbb{U}$  has continuous sample paths, so does  $\mathbb{B}$ . □

**Exercise 1.** Graph the first few  $g_{nj}$ 's and  $h_{nj}$ 's.

**Exercise 2.** Justify the interchange of expectation and summation used in the proof. [Hint: use the Tonelli part of Fubini's theorem.]

**Exercise 3.** Let  $\mathbb{U}$  be a Brownian bridge process. For  $0 \leq t < \infty$  define a process  $\mathbb{B}$  by

$$\mathbb{B}(t) \equiv (1+t)\mathbb{U}\left(\frac{t}{1+t}\right).$$

Show that  $\mathbb{B}$  is a Brownian motion process on  $[0, \infty)$ .