

Statistics 521, Problem Set 9 Solutions

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1. PfS, Exercise 7.1.1, page 124:

(a) Show that $P(AB) = P(A)P(B)$ if and only if $\{\emptyset, A, A^c, \Omega\}$ and $\{\emptyset, B, B^c, \Omega\}$ are independent σ -fields.

(b) Show that A_1, \dots, A_n are independent if and only if (for each $k = 1, \dots, n$,

$$P(A_{i_1} \dots A_{i_k}) = \prod_{j=1}^k P(A_{i_j}) \quad \text{whenever } 1 \leq i_1 < \dots < i_k \leq n.$$

Solution: We will prove (b) first.

(b) Since A_1, \dots, A_n are independent if and only if the random variables $1_{A_1}, \dots, 1_{A_n}$ are independent, if and only if the σ -fields $\mathcal{F}(1_{A_1}), \dots, \mathcal{F}(1_{A_n})$ are independent, and of course these are just the σ -fields

$\mathcal{A}_1 \equiv \{\emptyset, A_1, A_1^c, \Omega\}, \dots, \mathcal{A}_n \equiv \{\emptyset, A_n, A_n^c, \Omega\}$.

It remains only to show that the σ -fields $\mathcal{A}_1, \dots, \mathcal{A}_n$ are independent if and only if for each $k = 1, \dots, n$

$$P(A_{i_1} \cap \dots \cap A_{i_k}) = \prod_{j=1}^k P(A_{i_j}) \tag{1}$$

whenever $1 \leq i_1 < \dots < i_k \leq n$. First suppose that the σ -fields $\mathcal{A}_1, \dots, \mathcal{A}_n$ are independent. Then by taking $\Omega \in \{\emptyset, A_j, A_j^c, \Omega\}$ for $j \in \{i_1, \dots, i_k\}^c$, we have, with $B_j = A_{i_m}$ if $j = i_m$, $B_j = \Omega$ if $j \in \{i_1, \dots, i_k\}^c$,

$$P(A_{i_1} \cap \dots \cap A_{i_k}) = P(\cap_{j=1}^n B_j) = \prod_{j=1}^n P(B_j) = P(A_{i_1}) \dots P(A_{i_k})$$

since $P(B_j) = P(\Omega) = 1$ for $j \in \{i_1, \dots, i_k\}^c$; i.e. (1) holds. Now suppose that (1) holds. In particular, this implies that

$$P(B_1 \cap \dots \cap B_n) = P(B_1) \dots P(B_n) \tag{2}$$

where each $B_i \in \{\emptyset, A_i, \Omega\}$, $i = 1, \dots, n$, since both sides are 0 if any $B_j = \emptyset$, and if $B_j = A_j$ for $j \in \{j_1, \dots, j_m\} \subset \{1, \dots, n\}$, $B_j = \Omega$ for $j \in \{j_1, \dots, j_m\}^c \subset \{1, \dots, n\}$, then (2) reduces to (1). Now consider replacing B_1 by the other remaining element of \mathcal{A}_1 , A_1^c : if we replace B_1 by A_1^c , then since $\Omega = A_1 + A_1^c$ it follows that $\Omega \cap \bigcap_{k=2}^n A_k = \bigcap_{k=1}^n A_k + A_1^c \bigcap_{k=2}^n A_k$ and hence the left side of (2) becomes

$$\begin{aligned} P(\bigcap_{k=2}^n B_k) - P(A_1 \cap \bigcap_{k=2}^n B_k) &= P(B_2) \cdots P(B_n) - P(A_1)P(B_2) \cdots P(B_n) \\ &= (1 - P(A_1))P(B_2) \cdots P(B_n) \\ &= P(A_1^c)P(B_2) \cdots P(B_n); \end{aligned}$$

thus we have proved that

$$P(C_1 \cap B_2 \cdots \cap B_n) = P(C_1)P(B_2) \cdots P(B_n) \quad (3)$$

for $C_1 \in \mathcal{A}_1$, and $B_j \in \{\emptyset, A_j, \Omega\}$ for $j = 2, \dots, n$. This is the first step of an induction argument. Now suppose that for some $k \in \{1, \dots, n\}$.

$$P(C_1 \cdots C_{k-1} \cap B_k \cdots B_n) = P(C_1) \cdots P(C_{k-1})P(B_k) \cdots P(B_n) \quad (4)$$

for all $C_i \in \mathcal{A}_i$, $i = 1, \dots, k-1$, $B_i \in \{\emptyset, A_i, \Omega\}$, $i = k, \dots, n$. Since $\Omega = A_k + A_k^c$ it follows that

$$\Omega \cap \bigcap_{j=1}^{k-1} C_j \cap \bigcap_{j=k+1}^n B_j = A_k \cap \bigcap_{j=1}^{k-1} C_j \cap \bigcap_{j=k+1}^n B_j + A_k^c \cap \bigcap_{j=1}^{k-1} C_j \cap \bigcap_{j=k+1}^n B_j.$$

Thus upon replacing B_k by A_k^c on the left side of (4), we see that we have, since both $\Omega, A_k \in \{\emptyset, A_k, \Omega\}$,

$$\begin{aligned} &P(\bigcap_{j=1}^{k-1} C_j \cap A_k^c \cap \bigcap_{j=k+1}^n B_j) \\ &= P(\bigcap_{j=1}^{k-1} C_j \cap \Omega \cap \bigcap_{j=k+1}^n B_j) - P(\bigcap_{j=1}^{k-1} C_j \cap A_k \cap \bigcap_{j=k+1}^n B_j) \\ &= \prod_{j=1}^{k-1} P(C_j)P(\Omega) \prod_{j=k+1}^n P(B_j) - \prod_{j=1}^{k-1} P(C_j)P(A_k) \prod_{j=k+1}^n P(B_j) \\ &= \prod_{j=1}^{k-1} P(C_j) \cdot (1 - P(A_k)) \cdot \prod_{j=k+1}^n P(B_j) \\ &= \prod_{j=1}^{k-1} P(C_j) \cdot P(A_k^c) \cdot \prod_{j=k+1}^n P(B_j). \end{aligned}$$

Hence we have proved that (4) implies that

$$P(C_1 \cdots C_k B_{k+1} \cdots B_n) = P(C_1) \cdots P(C_k) P(B_{k+1}) \cdots P(B_n) \quad (5)$$

for all $C_i \in \mathcal{A}_i$, $i = 1, \dots, k$, $B_i \in \{\emptyset, \mathcal{A}_i, \Omega\}$, $i = k + 1, \dots, n$. It then follows by induction that

$$P(C_1 \cdots C_n) = P(C_1) \cdots P(C_n) \quad (6)$$

for all $C_i \in \mathcal{A}_i$, $i = 1, \dots, n$; i.e. the σ -fields \mathcal{A}_i , $i = 1, \dots, n$ are independent.

(a) This follows immediately from (b) with $n = 2$.

2. Give an example of two collections of sets \mathcal{A}_1 and \mathcal{A}_2 that are independent but the generated σ -fields are not independent.

Solution: One example of this goes as follows: let $\Omega = \{1, 2, 3, 4\}$, and let $\mathcal{A} = 2^\Omega$. Suppose that $P(\{\omega\}) = 1/4$ for each $\omega \in \Omega$. Suppose that $\mathcal{A}_1 = \{\{1, 2\}\}$ and $\mathcal{A}_2 = \{\{2, 3\}, \{2, 4\}\}$. For $A_1 \in \mathcal{A}_1$, $A_2 \in \mathcal{A}_2$, we have

$$P(A_1 \cap A_2) = P(\{2\}) = 1/4 = (1/2)(1/2) = P(A_1)P(A_2)$$

so that \mathcal{A}_1 , \mathcal{A}_2 are independent classes. Then we have $\{2\} \in \sigma[\mathcal{A}_2]$ (since $\{2\} = \{2, 3\} \cap \{2, 4\}$) and $\{1, 2\} \in \sigma[\mathcal{A}_1]$, but

$$P(\{1, 2\} \cap \{2\}) = P(\{2\}) = 1/4 \neq 1/8 = P(\{1, 2\})P(\{2\}).$$

Thus $\sigma[\mathcal{A}_1]$ and $\sigma[\mathcal{A}_2]$ are *not* independent classes. The difficulty here is that the class \mathcal{A}_2 is not a $\bar{\pi}$ -system.

3. Show that if X_n is any sequence of random variables, there are constants $c_n \rightarrow \infty$ so that $X_n/c_n \rightarrow_{a.s.} 0$.

Solution: Define $F_n(x) \equiv P(|X_n| \leq x)$, the distribution function of $|X_n|$. Set $b_n = F_n^{-1}(1 - n^{-2})$ for $n = 1, 2, \dots$, and let $\{a_n\}$ be any sequence with $a_n \rightarrow \infty$. Let $c_n = a_n b_n$. Then, for any $\epsilon > 0$ we have $\epsilon a_n \geq 1$ for $n \geq N_\epsilon$ and hence, using $F_n \circ F_n^{-1}(t) \geq t$ for all $0 < t < 1$,

$$\begin{aligned} P(|X_n| > \epsilon c_n) &= P(|X_n| > \epsilon a_n b_n) \\ &\leq P(|X_n| > b_n) \\ &= 1 - F_n(b_n) = 1 - F_n(F_n^{-1}(1 - n^{-2})) \\ &\leq n^{-2}, \quad n \geq N_\epsilon. \end{aligned}$$

Hence by the first Borel-Cantelli lemma, $P(|X_n| > \epsilon c_n \text{ i.o.}) = 0$ for every $\epsilon > 0$; that is, $X_n/c_n \rightarrow_{a.s.} 0$.

4. Show that if $P(A_n) \rightarrow 0$ and $\sum_{n=1}^{\infty} P(A_n \cap A_{n+1}^c) < \infty$, then $P(A_n \text{ i.o.}) = 0$.

Solution 1: Let $B_n \equiv A_n \cap A_{n+1}^c$. Then by the first Borel-Cantelli lemma, $P(\limsup_n B_n) = P(B_n \text{ i.o.}) = 0$. Furthermore, since $P(A_n) \rightarrow 0$ it follows that

$$P(\limsup_n A_n^c) = \lim_{n \rightarrow \infty} P(\cup_{m \geq n} A_m^c) \geq \lim_{n \rightarrow \infty} P(A_n^c) = 1.$$

Suppose that the following claim holds:

Claim: $\limsup_n A_n \cap \limsup_n A_n^c \subset B_n$.

Then

$$\begin{aligned} P(\limsup_n A_n) &= P(\limsup_n A_n \cap \limsup_n A_n^c) + P(\limsup_n A_n \cap [\limsup_n A_n^c]^c) \\ &\leq P(\limsup_n B_n) + P([\limsup_n A_n^c]^c) = 0 + 0 = 0. \end{aligned}$$

To prove the claim, let $\omega \in \limsup_n A_n \cap (\limsup_n A_n^c)$. Fix $N \geq 1$. Then there exists an $m \geq N$ such that $\omega \in A_m$. Now let $t \equiv \inf\{k > m : \omega \in A_k^c\}$. Such a t exists since $\omega \in \limsup_n A_n^c$. But then $\omega \in A_{t-1} \cap A_t^c$, and hence

$$\omega \in \cap_{n=1}^N \cup_{m=n}^{\infty} (A_m \cap A_{m+1}^c).$$

Since this holds for any N we have $\omega \in \cap_{n=1}^{\infty} \cup_{m=n}^{\infty} (A_m \cap A_{m+1}^c) = \limsup_n B_n$.

Solution 2: First we write

$$[A_n \text{ i.o.}] = \lim_n \cup_{m=n}^{\infty} A_m = \lim_n \lim_N \cup_{m=n}^N A_m$$

and hence

$$P[A_n \text{ i.o.}] = \lim_n \lim_N P(\cup_{m=n}^N A_m).$$

Now write

$$\cup_{m=n}^N A_m = (\cup_{m=n}^{N-1} (A_m \cap A_{m+1}^c)) \cup A_N,$$

and hence

$$P(\cup_{m=n}^N A_m) \leq P(\cup_{m=n}^{N-1} (A_m \cap A_{m+1}^c)) + P(A_N).$$

This yields

$$\begin{aligned} P[A_n \text{ i.o.}] &= \lim_n \lim_N P(\cup_{m=n}^N A_m) \\ &\leq \lim_n \lim_N \{P(\cup_{m=n}^{N-1} (A_m \cap A_{m+1}^c)) + P(A_N)\} \\ &\leq \lim_n \sum_{m=n}^{\infty} P(A_m \cap A_{m+1}^c) + \lim_{N \rightarrow \infty} P(A_N) \\ &= 0 + 0 = 0 \end{aligned}$$

since $\sum_{m=1}^{\infty} P(A_m \cap A_{m+1}^c) < \infty$ and $P(A_n) \rightarrow 0$.

5. Let X_1, X_2, \dots be independent. Show that $\sup_{n \geq 1} X_n < \infty$ almost surely if and only if $\sum_{n=1}^{\infty} P(X_n > M) < \infty$ for some $M < \infty$.

Solution: Suppose that $\sum_n P(X_n > M) < \infty$ for some $M < \infty$. Then by the first Borel-Cantelli lemma, $P(X_n > M \text{ i.o.}) = 0$; i.e. for $n \geq N_\omega$ we have $X_n(\omega) \leq M$. Thus

$$\sup_n X_n(\omega) \leq \left(\max_{1 \leq k < N_\omega} X_k \right) \vee M < \infty.$$

Now suppose that $\sup X_n < \infty$ almost surely. If $\sum_n P(X_n > M) = \infty$ for every $M < \infty$, then, by the second Borel-Cantelli lemma, $P(X_n > M \text{ i.o.}) = 1$ for every M ; i.e. $\limsup_{n \rightarrow \infty} X_n \geq M$ a.s. for every $M > 0$, and this implies, by taking a sequence $M_k \nearrow \infty$, that $\limsup_{n \rightarrow \infty} X_n = \infty$ a.s., which contradicts $\sup X_n < \infty$ almost surely. We therefore conclude that $\sum_n P(X_n > M) < \infty$ for some $M < \infty$.

6. Suppose that X_1, X_2, \dots are independent with $P(X_n > x) = x^{-r}$ for all $x \geq 1$ and $n = 1, 2, \dots$ with $r > 0$. Show that $\limsup_{n \rightarrow \infty} (\log X_n) / \log n = c$ almost surely for some number c , and find c .

Solution: Let $c > 0$. Then

$$P(\log X_n > c \log n) = P(X_n > n^c) = n^{-cr}.$$

Hence by the first and second Borel-Cantelli lemmas

$$P(\log X_n > c \log n \text{ i.o.}) = \begin{cases} 0 & \text{if } c > 1/r \\ 1 & \text{if } c \leq 1/r \end{cases} .$$

Hence

$$\limsup_{n \rightarrow \infty} (\log X_n) / (\log n) = 1/r \quad \text{almost surely.}$$