Statistics 521, Problem Set 8 Solutions

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Due: Wednesday, November 27, 2019

1. PfS Exercise 6.4.3, page 114. Prove just the parts of these formulas involving F, not the parts involving F^{-1}. You may also use Fubini's theorem directly. That is, show that:
   (i) If $X \ge 0$ has d.f. F, then
   $\int_0^\infty P(X > x)dx = E(X) = \int_0^\infty (1 - F(x))dx$.
   (ii) If $E|X| < \infty$ then
   $E(X) = -\int_{-\infty}^0 F(x)dx + \int_0^\infty (1 - F(x))dx$.
   (iii) Let $r > 0$. If $X \ge 0$, then
   $\int_0^\infty P(X^r > x)dx = E(X^r) = \int_0^\infty rx^{r-1}(1 - F(x))dx$.

   Solution: (11): Since $X \ge 0$ we may write $X = \int_0^X dt$ to obtain

$$\begin{aligned} E(X) &= \int_\Omega X dP = \int_\Omega \int_0^X dt dP \\ &= \int_\Omega \int_0^\infty 1_{\{0 \le t < X\}} dt dP \\ &= \int_0^\infty \int_\Omega 1_{\{0 \le t < X\}} dP dt \qquad \text{by Fubini's theorem} \\ &= \int_0^\infty P(X > t)dt = \int_0^\infty (1 - F(t))dt\,. \end{aligned}$$

   If either side is $+\infty$, then both sides are $+\infty$.
   (12): Since $X = X^+ - X^-$ with $X^+, X^- \ge 0$, it follows from (11) applied to X^+ and X^- (and assuming that $E|X| = EX^+ + EX^- < \infty$)

that

$$\begin{aligned}
E(X) &= E(X^+) - E(X^-) \\
&= \int_0^\infty P(X^+ > t)dt - \int_0^\infty P(X^- > t)dt \\
&= \int_0^\infty P(X > t)dt - \int_0^\infty P(-X > t)dt \\
&= \int_0^\infty (1 - F(t))dt - \int_0^\infty P(X < -t)dt \\
&= \int_0^\infty (1 - F(t))dt - \int_0^\infty F(-t-)dt \\
&= \int_0^\infty (1 - F(t))dt - \int_{-\infty}^0 F(t-)dt \\
&= \int_0^\infty (1 - F(t))dt - \int_{-\infty}^0 F(t)dt
\end{aligned}$$

where the last equality follows by a change of variables and the fact that $F(t)$ differs from $F(t-)$ on a set which has Lebesgue measure at most zero.
(13): Since $X \ge 0$ we may write $X^r = \int_0^X rt^{r-1}dt$ to obtain

$$\begin{aligned}
E(X^r) &= \int_\Omega X dP = \int_\Omega \int_0^X rt^{r-1}dtdP = \int_\Omega \int_0^\infty 1_{\{0 \le t < X\}} rt^{r-1}dtdP \\
&= \int_0^\infty \int_\Omega 1_{\{0 \le t < X\}} dP rt^{r-1}dt \\
&\qquad \text{by the Tonelli part of the Fubini-Tonelli theorem} \\
&= \int_0^\infty rt^{r-1}P(X > t)dt = \int_0^\infty rt^{r-1}(1 - F(t))dt\,.
\end{aligned}$$

If either side is $+\infty$, then both sides are $+\infty$.

2. Prove the two formulas in (17), PfS page 113: if $X \ge 0$ is integer valued, then $E(X) = \sum_{k=1}^\infty P(X \ge k)$ and $E(X^2) = \sum_{k=1}^\infty (2k-1)P(X \ge k)$.

   Solution: First note that if $X \ge 0$ is integer-valued, then the distribution function F of X is constant between integers, and $P(X > x) =$

$1 - F(x) = P(X > k)$ for $k \le x < k+1$. Thus from (7.4.11) we find that

$$\begin{aligned}
E(X) &= \int_0^\infty (1 - F(x))dx \\
&= \sum_{k=0}^\infty \int_{[k,k+1)} (1 - F(x))dx \\
&= \sum_{k=0}^\infty (1 - F(k)) \int_{[k,k+1)} dx \\
&= \sum_{k=0}^\infty P(X > k) = \sum_{k=0}^\infty P(X \ge k+1) \\
&= \sum_{m=1}^\infty P(X \ge m)\,.
\end{aligned}$$

Similarly, from (7.4.13) with $r = 2$,

$$\begin{aligned}
E(X^2) &= \int_0^\infty 2x(1 - F(x))dx \\
&= \sum_{k=0}^\infty \int_{[k,k+1)} 2x(1 - F(x))dx \\
&= \sum_{k=0}^\infty P(X > k) \int_{[k,k+1)} 2xdx \\
&= \sum_{k=0}^\infty P(X > k)(2k+1)
\end{aligned}$$

since

$$\int_k^{k+1} 2xdx = x^2\Big|_k^{k+1} = 2k + 1\,.$$

3. PfS Exercise 4.9, page 114: For any distribution function F on $\mathbb{R}$ we have $\int \{F(x + \theta) - F(x)\}dx = \theta$ for each $\theta \ge 0$.

   Solution: First note that

$$F(x + \theta) - F(x) = \int_{(x,x+\theta]} dF(y) = \int_{-\infty}^\infty 1_{(x,x+\theta]}(y)dF(y)$$

where the integrand, $h(x,y) \equiv 1_{(x,x+\theta]}(y) \geq 0$ for all x, y. Thus by the Tonelli part of the Fubini-Tonelli theorem

$$\begin{aligned}
\int \{F(x+\theta) - F(x)\}dx &= \int \left\{ \int_{-\infty}^{\infty} 1_{(x,x+\theta]}(y) dF(y) \right\} dx \\
&= \int_{-\infty}^{\infty} \left\{ \int 1_{(x,x+\theta]}(y) dx \right\} dF(y) \\
&= \int_{-\infty}^{\infty} \left\{ \int 1_{[y-\theta,y)}(x) dx \right\} dF(y) \\
&= \int_{-\infty}^{\infty} \{\theta\}\, dF(y) = \theta \cdot 1 = \theta.
\end{aligned}$$

4. PfS, Exercise 5.1.4, page 91.
   Let $\mathcal{X} = [0,1]$, $\mathcal{Y} = (1,\infty)$ both equipped with the Borel sets and Lebesgue measure. Let $f(x,y) = e^{-xy} - 2e^{-2xy}$. Show that
   (i) $\int_0^1 (\int_1^\infty f(x,y)dy)dx = \int_0^1 x^{-1}(e^{-x} - e^{-2x})dx$ exists and is > 0.
   (ii) $\int_1^\infty (\int_0^1 f(x,y)dx)dy = \int_1^\infty y^{-1}(e^{-2y} - e^{-y})dy$ exists and is < 0.

   Solution: First,

$$\int_0^1 \left(\int_1^\infty f(x,y)dy \right) dx = \int_0^1 x^{-1} \left(e^{-x} - e^{-2x} \right) dx$$

   by an easy calculation of the inner integral. Note that the function $x^{-1}(e^{-x} - e^{-2x})$ converges to 1 as $x \downarrow 0$, and is continuous elsewhere, hence is uniformly continuous and uniformly bounded on $[0,1]$. Since it is strictly positive and bounded, the last integral exists and is positive. On the other hand,

$$\int_1^\infty \left(\int_0^1 f(x,y)dx \right) dy = \int_1^\infty y^{-1}(e^{-2y} - e^{-y})dy$$

   again by an easy calculation of the inner integral. Now the integrand is negative (since $e^{-2y} < e^{-y}$ for all $y > 0$), bounded and the two integrals $\int_1^\infty y^{-1}e^{-2y}dy$ and $\int_1^\infty y^{-1}e^{-y}dy$ both are clearly finite. Thus the second iterated integral exists and is strictly negative. Letting $Ei(z) \equiv -\int_z^\infty t^{-1}e^{-t}dt$, the exponential integral function, it is easily shown that

$$\begin{aligned}\int_0^1 \left(\int_1^\infty f(x,y)dy\right) dx &= \int_0^1 x^{-1}\left(e^{-x} - e^{-2x}\right) dx \\ &= -Ei(-2) + Ei(-1) + \log(2) \\ &\doteq 0.522664...,\end{aligned}$$

while

$$\begin{aligned}\int_1^\infty \left(\int_0^1 f(x,y)dx\right) dy &= \int_1^\infty y^{-1}(e^{-2y} - e^{-y})dy \\ &= -Ei(-2) + Ei(-1) \\ &\doteq -0.170483....\end{aligned}$$

Of course the difficulty here is that

$$\int_1^\infty \left(\int_0^1 |f(x,y)|dx\right) dy = \int_0^1 \left(\int_1^\infty |f(x,y)|dy\right) dx = \infty\,.$$

In fact

$$\begin{aligned}\int_0^1 \left(\int_1^\infty f^+(x,y)dx\right) dy &= \int_0^1 \left(\int_{x^{-1}\log 2}^\infty f(x,y)dy\right) dx \\ &= \int_0^1 x^{-1}(e^{-\log 2} - e^{-2\log 2})dx \\ &= \frac{1}{4}\int_0^1 x^{-1}dx = +\infty,\end{aligned}$$

and similarly

$$\begin{aligned}\int_0^1 \left(\int_1^\infty f^-(x,y)dx\right) dy &= \int_0^1 \left(\int_0^{x^{-1}\log 2} f(x,y)dy\right) dx \\ &= \int_0^1 x^{-1}((1 - e^{-2\log 2}) - (1 - e^{-\log 2}))dx \\ &= \frac{1}{4}\int_0^1 x^{-1}dx = +\infty.\end{aligned}$$

See Figures 1-3.

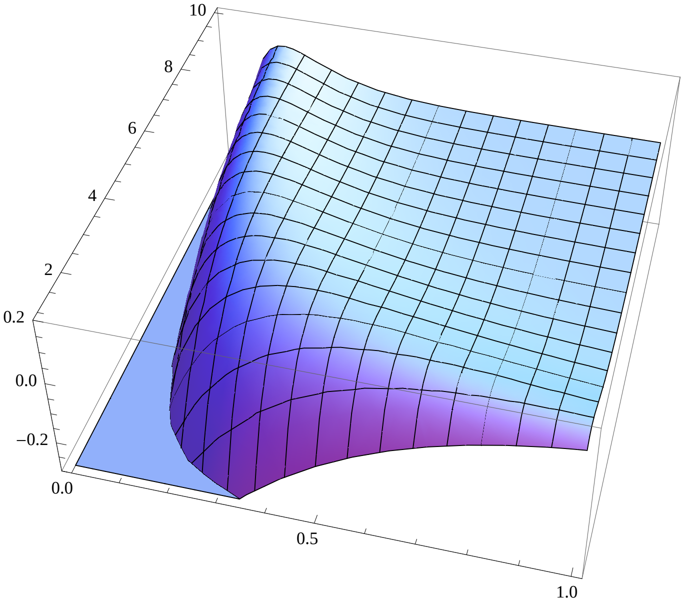

Figure 1: The function $f(x, y)$.

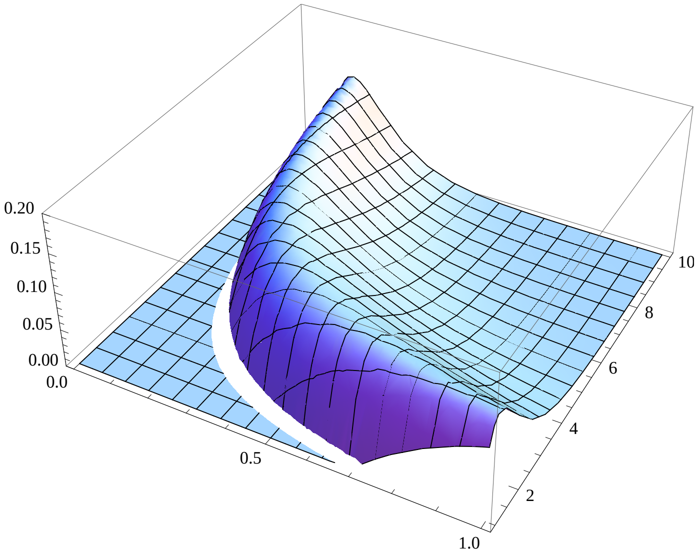

Figure 2: The function $f^{+}(x, y)$.

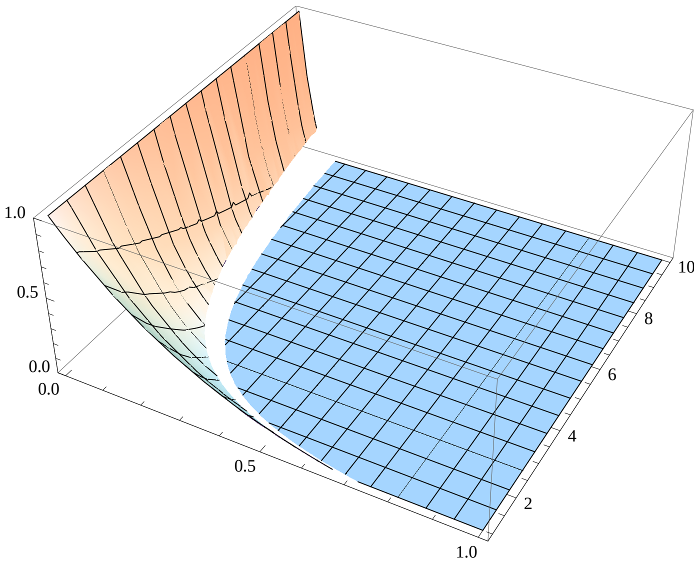

Figure 3: The function $f^{-}(x, y)$.