Statistics 521, Problem Set 7 Solutions

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1. PfS, Exercise 4.1.2, page 67: Identify ϕ^+, ϕ^-, $|\phi|$ and $|\phi|(\Omega)$ in the context of the prototypical situation of example 4.1.1, page 66: in particular $\phi(A) = \int_A X d\mu$ where X is measurable. Be sure to specify Ω^+ and Ω^- as in the Jordan - Hahn decomposition.

   Solution: I claim that

$$\begin{aligned}
\phi^+(A) &= \int_A X^+ d\mu = \phi(A\Omega^+) \ \text{ with } \ \Omega^+ = \{\omega : X(\omega) \geq 0\}, \\
\phi^-(A) &= \int_A X^- d\mu = -\phi(A\Omega^-) \ \text{ with } \ \Omega^- = \{\omega : X(\omega) < 0\}, \\
|\phi|(A) &= \int_A |X| d\mu, \qquad \text{and} \\
|\phi|(\Omega) &= \int |X| d\mu.
\end{aligned}$$

   To see this, note that Ω^+, Ω^- are, respectively, positivity, negativity sets for ϕ since

$$\begin{aligned}
\phi(A) &= \int_A X d\mu \geq 0 \qquad \text{for all events } \ A \subset \Omega^+, \\
\phi(A) &= \int_A X d\mu \leq 0 \qquad \text{for all events } \ A \subset \Omega^-.
\end{aligned}$$

   Furthermore, if $\tilde{\Omega}^+$, $\tilde{\Omega}^-$ denote the decomposition guaranteed by the Jordan-Hahn theorem 1.1, then

$$\begin{aligned}
\phi(\Omega^+ \setminus \tilde{\Omega}^+) = \phi(\Omega^+ \cap \tilde{\Omega}^-) = 0, \qquad \text{and} \\
\phi(\tilde{\Omega}^+ \setminus \Omega^+) = \phi(\tilde{\Omega}^+ \cap \Omega^-) = 0,
\end{aligned}$$

   where the zeroes follow by using the definitions of Ω^+, Ω^-, $\tilde{\Omega}^+$, $\tilde{\Omega}^-$. Thus

$$|\phi|(\Omega^+ \Delta \tilde{\Omega}^+) = 0;$$

i.e. $\Omega^+ = [X \geq 0]$ differs from $\tilde{\Omega}^+$ by (at most) a set of $|\phi|-$measure 0. Finally we have

$$\begin{aligned} |\phi|(A) &= \phi^+(A) + \phi^-(A) = \int_A X^+ d\mu + \int_A X^- d\mu \\ &= \int_A |X| d\mu, \end{aligned}$$

and $|\phi|(\Omega) = \int_\Omega |X| d\mu$.

2. Let μ be a sigma-finite measure and let ν be a finite measure on $(\Omega, \mathcal{A})$. Set $\phi = \mu - \nu$; i.e. define $\phi : \mathcal{A} \to (-\infty, \infty]$ by $\phi(A) = \mu(A) - \nu(A)$.
(a) Show that ϕ is a signed measure.
(b) Show that

$$\phi(A) = \int_A (f - g) d(\mu + \nu)$$

for some measurable functions f and g, $g \in \mathcal{L}_1(\mu + \nu)$. Thus ϕ can be written in the canonical form of the signed measure discussed in example 4.1.1 and 1 above.
(c) Apply the results of PfS Exercise 4.1.2, page 67, to ϕ: compute ϕ^+, ϕ^-, $|\phi|$, and $|\phi|(\Omega)$, assuming for the latter that μ is also a finite measure.

Solution: (a) First $\phi(\emptyset) = \mu(\emptyset) - \nu(\emptyset) = 0 - 0 = 0$,; next, for $A \in \mathcal{A}$ we have $\phi(A) = \mu(A) - \nu(A) \in (-\infty, \infty]$ since $\nu(\Omega) < \infty$ and $\mu(A), \nu(A) \geq 0$; finally, for any collection of disjoint sets $A_n \in \mathcal{A}$,

$$\begin{aligned} \phi(\sum A_n) &= \mu(\sum A_n) - \nu(\sum A_n) = \sum \mu(A_n) - \sum \nu(A_n) \\ &= \sum(\mu(A_n) - \nu(A_n)) = \sum \phi(A_n) \,. \end{aligned}$$

Thus ϕ is a signed measure.
(b) Now $\mu \ll \mu+\nu$ and $\nu \ll \mu+\nu$, so by the Radon-Nikodym theorem, $f = \frac{d\mu}{d(\mu+\nu)}$ and $g = \frac{d\nu}{d(\mu+\nu)}$ exist, are measurable, and, for $A \in \mathcal{A}$,

$$\mu(A) = \int_A f d(\mu + \nu), \qquad \nu(A) = \int_A g d(\mu + \nu) \,.$$

Hence we have

$$\phi(A) = \mu(A) - \nu(A) = \int_A (f - g) d(\mu + \nu) \,.$$

(c) Note that in part (b) we have written ϕ in the form of the signed measure treated in problem #1 with X replaced by $(f-g)$ and μ replaced by $\mu+\nu$. Hence it follows that

$$\phi^+(A) = \int_A (f-g)^+ d(\mu+\nu)\,, \qquad \phi^-(A) = \int_A (f-g)^- d(\mu+\nu)$$

and

$$|\phi|(A) = \int_A |f-g| d(\mu+\nu)\,.$$

Thus we have $|\phi|(\Omega) = \int_\Omega |f-g|d(\mu+\nu)$.

3. For probability measures P and Q on $(\Omega, \mathcal{A})$, define

$$d_{TV}(P,Q) = \sup_{A\in\mathcal{A}} |P(A) - Q(A)|\,.$$

We showed in problem 3, problem set #6 that $d_{TV}(P,Q) = (1/2)\int |p-q|d\mu$ for any measure μ dominating both P and Q; i.e. $P << \mu$, $Q << \mu$.
(a) Show that $d_{TV}(P,Q)$ does not depend on the choice of μ.
(b) Use the results of problem 1, part (c) to show that $d_{TV}(P,Q) = (1/2)|P-Q|(\Omega)$.

Solution: (a) This is actually pretty easy given the equality $d_{TV}(P,Q) \equiv \sup_{A\in\mathcal{A}} |P(A)-Q(A)| = (1/2)\int |p_\mu - q_\mu| d\mu$ for any measure μ dominating both P and Q: if μ and ν are two such measures then we have

$$\frac{1}{2}\int |p_\mu - q_\mu| d\mu = \sup_{A\in\mathcal{A}} |P(A)-Q(A)| = \frac{1}{2}\int |p_\nu - q_\nu| d\nu\,.$$

On the other hand, another proof which does not use this identity goes as follows. Since $\mu \ll \mu+\nu$ and $\nu \ll \mu+\nu$, we know that the Radon-Nikodym derivatives $f = d\mu/d(\mu+\nu)$ and $g = d\nu/d(\mu+\nu)$ exist. Furthermore, $P \ll \mu+\nu$ and $Q \ll \mu+\nu$. Hence

$$\begin{aligned} p_{\mu+\nu} &= \frac{dP}{d(\mu+\nu)} = \frac{dP}{d\mu}\frac{d\mu}{d(\mu+\nu)} = p_\mu f \\ &= \frac{dP}{d(\mu+\nu)} = \frac{dP}{d\nu}\frac{d\nu}{d(\mu+\nu)} = p_\nu g\,; \end{aligned}$$

similarly, $q_{\mu+\nu} = q_\mu f = q_\nu g$. Hence it follows that

$$\begin{aligned} \int |p_\mu - q_\mu| d\mu &= \int |p_\mu - q_\mu| f d(\mu+\nu) \\ &= \int |p_\mu f - q_\mu f| d(\mu+\nu) \\ &= \int |p_{\mu+\nu} - q_{\mu+\nu}| d(\mu+\nu) \\ &= \int |p_\nu - q_\nu| d\nu \end{aligned}$$

where the last inequality follows from the previous chain of equalities with μ replaced by ν and f replaced by g.
(b) From problem #1, part C, for any measure μ dominating both P and Q (e.g. $\mu = P+Q$) we have $|P-Q|(\Omega) = \int |p-q| d\mu = 2d_{TV}(P,Q)$.

4. Suppose that ϕ is a sigma-finite signed measure and $X \in L_1(|\phi|)$. The integral $\int X d\phi$ is defined by

$$\int X d\phi = \int X d\phi^+ - \int X d\phi^- .$$

Show that $|\int X d\phi| \le \int |X| d|\phi|$.

Solution: Note that

$$\begin{aligned} \int_\Omega X d\phi &= \int X d\phi^+ - \int X d\phi^- \\ &= \int X^+ d\phi^+ - \int X^- d\phi^+ - \left(\int X^+ d\phi^- - \int X^- d\phi^-\right) \end{aligned}$$

where all integrals, $\int X^+ d\phi^+$, $\int X^- d\phi^+$, $\int X^+ d\phi^-$, and $\int X^- d\phi^-$ are ≥ 0. Thus it follows that

$$\begin{aligned} \left| \int_\Omega X d\phi \right| &\le \int X^+ d\phi^+ + \int X^- d\phi^+ \\ &\quad + \int X^+ d\phi^- + \int X^- d\phi^- \\ &= \int |X| d\phi^+ + \int |X| d\phi^- \\ &= \int |X| d|\phi| . \end{aligned}$$

5. PfS, Exercise 4.2.3, page 73: Flip a coin. If heads results, let X be a Uniform$(0,1)$ random variable; if tails results, let X be a Poisson(λ) random variable. The resulting distribution of X on $\mathbb{R}$ is labeled ϕ.
(a) Let μ denote Lebesgue measure on $\mathbb{R}$. Find the Lebesgue decomposition of ϕ with respect to μ; that is, write $\phi = \phi_{ac} + \phi_s$.
(b) Let ν be counting measure on $\{0, 1, 2, \ldots\}$. Find the Lebesgue decomposition of ϕ with respect to ν.

Solution: First let $Y \sim$ Bernoulli$(1/2)$ denote the coin toss variable with $P(Y=1) = 1/2 = P(Y=0)$. Then

$$P([X \le x] \cap [Y=1]) = (1/2)\{(x \vee 0) \wedge 1\},$$
$$P([X \le x] \cap [Y=0]) = (1/2)\sum_{k=0}^{[x]} \exp(-\lambda)\frac{\lambda^k}{k!},$$

for $x \in \mathbb{R}$, and ϕ is the measure on $\mathbb{R}$ corresponding to the distribution function given by

$$F(x) = P(X \le x) = (1/2)\{(x \vee 0) \wedge 1\} + (1/2)\sum_{k=0}^{[x]} \exp(-\lambda)\frac{\lambda^k}{k!}.$$

(a) If μ is Lebesgue measure on $\mathbb{R}$, then the Poisson part of ϕ is singular with respect to μ, and the Uniform$(0,1)$ part is absolutely continuous with respect to μ. Thus with

$$\phi_{ac}(A) = (1/2)\mu(A \cap [0,1]) = \int_A \frac{1}{2}1_{(0,1)}(z)d\mu(z) = \int_A \frac{1}{2}1_{(0,1)}(z)dz,$$
$$\phi_s(A) \equiv \frac{1}{2}\sum_{k \in A} \exp(-\lambda)\frac{\lambda^k}{k!},$$

we have $\phi(A) = \phi_{ac}(A) + \phi_s(A)$ for all Borel sets A. Note that with $D \equiv \{0, 1, 2, \ldots\}$ we have $\phi_{ac}(D) = 0$ and $\phi_s(D^c) = 0$.
(b) If μ is counting measure on $D = \{0, 1, 2, \ldots\}$, then the Uniform$(0,1)$ part of ϕ is singular with respect to μ, and the Poisson part of ϕ is absolutely continuous with respect to μ. Thus with

$$\phi_s(A) = (1/2)\mu(A \cap [0,1]) = \int_A \frac{1}{2}1_{(0,1)}(z)dz,$$
$$\phi_{ac}(A) \equiv \frac{1}{2}\sum_{k \in A} \exp(-\lambda)\frac{\lambda^k}{k!} = \int_A \frac{1}{2}e^{-\lambda}\frac{\lambda^k}{k!}d\mu(k),$$

we can write $\phi(A) = \phi_{ac}(A) + \phi_s(A)$. Note that now $\phi_{ac}(D^c) = 0$ while $\phi_s(D) = 0$.

6. PfS, Exercise 4.4.3, page 84: Let F be $\nearrow$, right-continuous and bounded on $\mathbb{R}$ with $F(-\infty) = 0$. Define μ_F via $\mu_F((a, b]) = F(b) - F(a)$ for all $a < b$. Show that $\mu_F << \lambda$ if and only if F is an absolutely continuous function on $\mathbb{R}$ (in the sense of PfS, Definition 4.4.2, page 80).

   Solution: First suppose $\mu_F << \lambda$. Then by Theorem 4.2.1, page 72 (the Radon-Nikodym theorem), for all Borel sets A we have

$$\mu_F(A) = \int_A Z_0 d\lambda.$$

   Since F is bounded and $F(-\infty) = 0$

$$\mu_F(\mathbb{R}) = \int_R Z_0 d\lambda < \infty, \qquad \text{and} \qquad Z_0 \geq 0.$$

   Thus $Z_0 \in \mathcal{L}_1(\lambda)$. By Theorem 3.2.5 (absolute continuity of the integral), for every $\epsilon > 0$ there exists a $\delta_\epsilon > 0$ such that $\lambda(A) < \delta_\epsilon$ implies $\mu_F(A) = \int_A Z_0 d\lambda < \epsilon$. Take $A = \sum_{k=1}^n (c_k, d_k]$ with $\lambda(A) = \sum_{k=1}^n (d_k - c_k) < \delta_\epsilon$. Then

$$\epsilon > \int_A Z_0 d\lambda = \mu_F(A) = \mu_F(\sum_{k=1}^n (c_k, d_k]) = \sum_{k=1}^n |F(d_k) - F(c_k)|;$$

   i.e. F is an absolutely continuous function on $\mathbb{R}$.

   Conversely, suppose F is an absolutely continuous function on $\mathbb{R}$ with $F(-\infty) = 0$, $F \nearrow$, and $F(\infty) < \infty$. Then, by Theorem 4.4.1 (the fundamental theorem of calculus), F' exists a.e. λ and

$$F(x) = F(x) - F(-\infty) = \int_{-\infty}^x F' d\lambda \qquad \text{for all } x \in \mathbb{R}.$$

   Thus if μ_F is the corresponding measure with $\mu_F((a, b]) = F(b) - F(a)$, it follows that for any Borel set A

$$\mu_F(A) = \int_A F' d\lambda.$$

   But then $\mu_F << \lambda$ by the Radon-Nikodym theorem 4.2.1.