Statistics 521, Problem Set 2 Solutions

Wellner; 10/15/2019

1. (Carried over from problem set # 1). PfS, Exercise 1.1.2, PfS, page 8. We always have $\mu(\liminf A_n) \le \liminf \mu(A_n)$, while $\limsup \mu(A_n) \le \mu(\limsup A_n)$ if $\mu(\Omega) < \infty$.

   Solution: First note that $\liminf A_n = \cup_{n=1}^{\infty} \cap_{k=n}^{\infty} A_k = \cup_{n=1}^{\infty} B_n$ where $B_n = \cap_{k=n}^{\infty} A_k$ is $\uparrow$ since $B_n = \cap_{k=n}^{\infty} A_k \subset \cap_{k=n+1}^{\infty} A_k = B_{n+1}$ for all n. Hence by Proposition 1.2(i),

$$\begin{aligned}
\mu(\liminf A_n) &= \mu(\cup_{n=1}^{\infty} B_n) \\
&= \lim_{n\to\infty} \mu(B_n) \\
&= \lim_{n\to\infty} \mu(\cap_{k=n}^{\infty} A_k) \\
&\le \lim_{n\to\infty} \inf_{m\ge n} \mu(A_m) \\
&= \liminf \mu(A_n)
\end{aligned}$$

   since $\cap_{k=n}^{\infty} A_k \subset A_m$ for each $m \ge n$ so that

$$\mu(\cap_{k=n}^{\infty} A_k) \le \mu(A_m)$$

   for each $m \ge n$ and also $\mu(\cap_{k=n}^{\infty} A_k) \le \inf_{m\ge n} \mu(A_m)$.
   Similarly, $\limsup A_n = \cap_{n=1}^{\infty} \cup_{k=n}^{\infty} A_k = \cap_{n=1}^{\infty} B_n$ where $B_n = \cup_{k=n}^{\infty} A_k$ is $\downarrow$ since $B_n = \cup_{k=n}^{\infty} A_k \supset \cup_{k=n+1}^{\infty} A_k = B_{n+1}$. Thus by Proposition 1.1.2(ii), if $\mu(\Omega) < \infty$,

$$\begin{aligned}
\mu(\limsup A_n) &= \mu(\cap_{n=1}^{\infty} B_n) \\
&= \lim_{n\to\infty} \mu(B_n) \\
&= \lim_{n\to\infty} \mu(\cup_{k=n}^{\infty} A_k) \\
&\ge \lim_{n\to\infty} \sup_{m\ge n} \mu(A_m) \\
&= \limsup \mu(A_n)
\end{aligned}$$

   since $\cup_{k=n}^{\infty} A_k \supset A_m$ for each $m \ge n$ so that

$$\mu(\cup_{k=n}^{\infty} A_k) \ge \mu(A_m)$$

   for each $m \ge n$ and also $\mu(\cup_{k=n}^{\infty} A_k) \ge \sup_{m\ge n} \mu(A_m)$.

2. PfS, Exercise 1.1.3, page 9 (and read the proof of the $\pi - \lambda$ theorem, Proposition 1.1.5, pages 9-10).
(a) The minimal $\lambda-$system generated by the class $\mathcal{D}$ is denoted by $\lambda[\mathcal{D}]$. Show that $\lambda[\mathcal{D}]$ is equal to the intersection of all $\lambda-$systems containing $\mathcal{D}$.
(b) A collection $\mathcal{A}$ of subsets of Ω is a $\sigma-$field if and only if it is both a $\pi-$system and a $\lambda-$system.
(c) Let $\mathcal{C}$ be a $\pi-$system and let $\mathcal{D}$ be a $\lambda-$system. Then $\mathcal{C} \subset \mathcal{D}$ implies that $\sigma[\mathcal{C}] \subset \mathcal{D}$.

Solution: (a) Let

$$\lambda[\mathcal{D}] \equiv \cap\{\mathcal{F}_\alpha : \ \mathcal{F}_\alpha \text{ is a } \lambda - \text{system with } \mathcal{D} \subset \mathcal{F}_\alpha\}.$$

Now $\Omega \in \mathcal{F}_\alpha$ for all α, so $\Omega \in \lambda[\mathcal{D}]$. Further, if $A, B \in \lambda[\mathcal{D}]$ with $B \subset A$, then $B, A \in \mathcal{F}_\alpha$ for all α and hence $A \setminus B \in \mathcal{F}_\alpha$ for all α, and hence $A \setminus B \in \lambda[\mathcal{D}]$. Finally, if $\{A_1, A_2, \ldots\}$ is an increasing family of sets in $\lambda[\mathcal{D}]$, then $\{A_1, A_2, \ldots\} \subset \mathcal{F}_\alpha$ for all α. Hence $\lim_n A_n \in \mathcal{F}_\alpha$ for all α, and hence $\lim_n A_n \in \mathcal{F}_\alpha$ for all α. Thus it follows that $\lim_n A_n \in \lambda[\mathcal{D}]$. Thus $\lambda[\mathcal{D}]$ is a $\lambda-$system. If $\mathcal{A}'$ is a $\lambda-$system such that $\mathcal{D} \subset \mathcal{A}'$, then $\mathcal{A} = \mathcal{F}_\alpha$ for some α, and hence $\lambda[\mathcal{D}] \subset \mathcal{A}'$; i.e. $\lambda[\mathcal{D}]$ is the minimal $\lambda-$system containing $\mathcal{D}$.
(b) Suppose first that $\mathcal{A}$ is a $\sigma-$field. Thus it is closed under countable intersections, and hence, in particular it is closed under finite intersections and is a $\pi-$system. To show that $\mathcal{A}$ is a $\lambda-$system, first consider $A, B \in \mathcal{A}$ with $A \subset B$. Since $\mathcal{A}$ is closed under complementation, $A^c \in \mathcal{A}$, and hence also $B \cap A^c = B \setminus A \in \mathcal{A}$. Also $\Omega \in \mathcal{A}$ since it is a $\sigma-$field. Finally, if $\{A_n\}$ is a sequence of sets in $\mathcal{A}$ with $A_n \uparrow A$, then $A = \lim_n A_n = \cup_{n=1}^\infty A_n \in \mathcal{A}$ since $\mathcal{A}$ is a $\sigma-$field. Thus $\mathcal{A}$ is a $\lambda-$system, and this completes the proof that a $\sigma-$field is both a $\pi-$system and a $\lambda-$system.

Now suppose that $\mathcal{A}$ is a $\pi-$system and a $\lambda-$system. Let $A \in \mathcal{A}$. Since $\mathcal{A}$ is a $\lambda-$system, $\Omega \in \mathcal{A}$. Since $A \subset \Omega$ and $\mathcal{A}$ is a $\lambda-$system, $A^c = \Omega \cap C^c = \Omega \setminus A \in \mathcal{A}$. To show that $\mathcal{A}$ is closed under countable unions, suppose that $\{A_n\}$ is a countable family of sets with $A_n \in \mathcal{A}$ for each n. Set $B_n \equiv \cup_{i=1}^n A_i$. Then $B_n \in \mathcal{A}$ for each n since $A, B \in \mathcal{A}$ implies that $A \cup B = (A^c \setminus B)^c \in \mathcal{A}$ since $\mathcal{A}$ is a $\pi-$ system and a $\lambda-$system implies that it is closed under intersections, complements,

and set differences. But since $\mathcal{A}$ is a $\lambda-$system this implies that $\cup_{n=1}^{\infty} = \lim_n \cup_{i=1}^{n} A_i = \lim_n \cup_{i=1}^{n} B_i = \lim_n B_n \in \mathcal{A}$, and hence $\mathcal{A}$ is closed under countable unions. Hence $\mathcal{A}$ is a $\sigma-$field.
(c) It is clear that $\lambda(\mathcal{C}) \subset \sigma[\mathcal{C}]$ (since there are fewer restrictions in defining a $\lambda-$system than a field; or from (b)). If we show that $\lambda[\mathcal{C}]$ is a $\pi-$system, then since $\sigma[\mathcal{C}]$ is a $\lambda-$system containing $\mathcal{C}$, it must also be the minimal $\lambda-$system containing $\mathcal{C}$ and hence $\sigma[\mathcal{C}] = \lambda[\mathcal{C}] \subset \lambda[\mathcal{D}] \subset \mathcal{D}$. Thus it suffices to show that $\lambda[\mathcal{C}]$ is a $\pi-$system.

We do this in two steps:
Step 1: Let

$$\mathcal{D}_1 = \{B \in \lambda(\mathcal{C}) : \quad B \cap C \in \lambda(\mathcal{C}) \text{ for all } \ C \in \mathcal{C}\}$$

where $\mathcal{C}$ is a $\pi-$system and $\lambda(\mathcal{C})$ is the smallest $\lambda-$system containing $\mathcal{C}$. To show that $\mathcal{D}_1$ is a $\lambda-$system we need to show that:
(i) $\Omega \in \mathcal{D}_1$.
(ii) If $D_n \in \mathcal{D}_1$, $D_n \uparrow$, then $\cup D_n \in \mathcal{D}_1$.
(iii) If $A, B \in \mathcal{D}_1$ with $A \subset B$, then $A \setminus B \in \mathcal{D}_1$.
Proof of (i): $\Omega \in \lambda(\mathcal{C})$ since it is a $\lambda-$system, so we have $\Omega \cap C = C \in \mathcal{C} \subset \lambda(\mathcal{C})$ for each $C \in \mathcal{C}$, and hence $\Omega \in \mathcal{D}_1$.
Proof of (ii): Suppose $D_1, D_2, \ldots \in \mathcal{D}_1$ and $D_n \uparrow$. Then we have $\cup_n D_n \in \lambda(\mathcal{D})$ (since each $D_n \in \lambda(\mathcal{D})$, a $\lambda-$system), and $(\cup_n D_n) \cap C = \cup_n (D_n \cap C) = \cup_n B_n \in \lambda(\mathcal{C})$ since $B_n \in \lambda(\mathcal{C})$ is $\uparrow$. Hence $\cup_n D_n \in \mathcal{D}_1$.
Proof of (iii): Suppose $A, B \in \mathcal{D}_1$ with $B \subset A$. Then $AC, BC, A, B \in \lambda(\mathcal{C})$ for all $C \in \mathcal{C}$, so

$$(A \setminus B) \cap C = (A \cap C) \setminus (B \cap C) \in \lambda(\mathcal{C})$$

for all $C \in \mathcal{C}$. Hence $A \setminus B \in \mathcal{D}_1$.

Step 2: Let

$$\mathcal{D}_2 \equiv \{A \in \lambda[\mathcal{C}] : B \cap A \in \lambda[\mathcal{C}], \text{ for all } B \in \lambda[\mathcal{C}]\}.$$

Step 1 showed that $\mathcal{D}_2$ contains $\mathcal{C}$. As in step 1, we can show that $\mathcal{D}_2$ inherits the $\lambda-$system structure form $\lambda[\mathcal{C}]$ and that therefore $\mathcal{D}_2 = \lambda[\mathcal{C}]$. But the fact that $\lambda[\mathcal{C}] = \mathcal{D}_2$ means that $\lambda[\mathcal{C}]$ is a $\pi-$system.

3. PfS, Exercise 1.2.1, page 15. Let $(\Omega, \mathcal{A}, \mu)$ denote a measure space.

Show that

$$\begin{aligned}\widehat{\mathcal{A}}_\mu &\equiv \{A : A_1 \subset A \subset A_2,\ A_1, A_2 \in \mathcal{A}, \mu(A_2 \setminus A_1) = 0\} \\ &= \{A \cup N : A \in \mathcal{A} \text{ and } N \subset (\text{some}\, B) \in \mathcal{A} \ \text{ with } \ \mu(B) = 0\} \\ &= \{A \triangle N : \quad A \in \mathcal{A}, \quad N \subset (\text{some}\, B) \in \mathcal{A} \ \text{ with } \ \mu(B) = 0\}\end{aligned}$$

and is a $\sigma-$field. Show that $(\Omega, \widehat{\mathcal{A}}_\mu, \widehat{\mu})$ is complete.

Solution: Let these three collections of sets be called $\widehat{\mathcal{A}}_1$, $\widehat{\mathcal{A}}_2$, and $\widehat{\mathcal{A}}_3$ respectively.
To show that $\widehat{\mathcal{A}}_1 \subset \widehat{\mathcal{A}}_2$, suppose that $D \in \widehat{\mathcal{A}}_1$. Then there exist sets $A_1, A_2 \in \mathcal{A}$ such that $A_1 \subset D \subset A_2$ and $\mu(A_2 \setminus A_1) = 0$. Let $B = A_2 \setminus A_1$; then $\mu(B) = 0$ and $B \in \mathcal{A}$. Furthemore $D = A_1 \cup N$ for some $N \subset B$. Hence with $A = A_1$ and $B = A_2 \setminus A_1$ we have $D = A \cup N$ where $A \in \mathcal{A}$ and $N \subset B$ with $\mu(B) = 0$. Thus $D \in \widehat{\mathcal{A}}_2$, and $\widehat{\mathcal{A}}_1 \subset \widehat{\mathcal{A}}_2$.
To show that $\widehat{\mathcal{A}}_2 \subset \widehat{\mathcal{A}}_3$, let $D \in \widehat{\mathcal{A}}_2$. Then $D = A \cup N$ with $A \in \mathcal{A}$, $N \subset B \in \mathcal{A}$ having $\mu(B) = 0$, so $A \subset D$. Let $N_1 \equiv D \setminus A = D \cap A^c = N \cap A^c \subset N \subset B$. Hence we have

$$\begin{aligned}A \triangle N_1 &= (A^c \cap (D \cap A^c)) \cup (A \cap (D \cap A^c)^c) \\ &= (D \cap A^c) \cup ((A \cap D^c) \cup A) \\ &= (D \cap A^c) \cup A = D \cup A = D\end{aligned}$$

and hence $D \in \widehat{\mathcal{A}}_3$.
To show that $\widehat{\mathcal{A}}_3 \subset \widehat{\mathcal{A}}_1$, let $D \in \widehat{\mathcal{A}}_3$. Then $D = A \triangle N$ with $A \in \mathcal{A}$ and $N \subset B \in \mathcal{A}$ with $\mu(B) = 0$. Take $A_1 = A \cap B^c$, $A_2 = A \cup B$. Then $A_1 \subset A \cap N^c \subset D \subset A \cup N \subset A_2$. Since $A_2 \setminus A_2 = (A \cup B) \cap (A \cap B^c)^c = A \cap (A^c \cup B) \cup B \cap A^c \cap B = (A \cap B) \cup (A^c \cap B) = B$ so that $\mu(A_2 \setminus A_1) = \mu(B) = 0$, it follows that $D \in \widehat{\mathcal{A}}_1$, and hence that $\widehat{\mathcal{A}}_3 \subset \widehat{\mathcal{A}}_1$.

Since we have shown that

$$\widehat{\mathcal{A}}_1 \subset \widehat{\mathcal{A}}_2 \subset \widehat{\mathcal{A}}_3 \subset \widehat{\mathcal{A}}_1\,,$$

it follows that $\widehat{\mathcal{A}}_1 = \widehat{\mathcal{A}}_2 = \widehat{\mathcal{A}}_3$.

To show that $\widehat{\mathcal{A}}_\mu$ is a $\sigma-$field:
Let $A \in \widehat{\mathcal{A}}_1$. Then $A_1 \subset A \subset A_2$, so that $A_2^c \subset A^c \subset A_1^c$ with $A_2^c, A_1^c \in$

$\mathcal{A}$, and where $A_1^c \setminus A_2^c = A_2 \setminus A_1$ and hence $\mu(A_1^c \setminus A_2^c) = \mu(A_2 \setminus A_1) = 0$. Hence $A^c \in \widehat{\mathcal{A}}_1 = \widehat{\mathcal{A}}_\mu$.
Let $D_1, \ldots, D_n \in \widehat{\mathcal{A}}_2$. Then $D_n = A_n \cup N_n$ with $A_n \in \mathcal{A}$, $N_n \subset B_n \in \mathcal{A}$ with $\mu(B_n) = 0$ for each n. Thus $\cup A_n \in \mathcal{A}$, $\cup B_n \in \mathcal{A}$, $\cup N_n \subset \cup B_n$, and $\mu(\cup B_n) \leq \sum \mu(B_n) = 0$. Hence $\cup D_n = (\cup A_n) \cup (\cup N_n) \in \widehat{\mathcal{A}}_2$. Hence $\widehat{\mathcal{A}}_\mu$ is a $\sigma-$field.

To show that $(\Omega, \widehat{\mathcal{A}}_\mu, \widehat{\mu})$ is complete, first let $A_1 \cup N_1 = A_2 \cup N_2$ with $A_1, A_2 \in \mathcal{A}$, $N_1 \subset B_1$, $N_2 \subset B_2$ with $\mu(B_1) = \mu(B_2) = 0$. By definition $\widehat{\mu}(A_1 \cup N_1) = \mu(A_1)$ $\widehat{\mu}(A_2 \cup N_2) = \mu(A_2)$. But $A_1 \subset A_1 \cup N_1 = A_2 \cup N_2 \subset A_2 \cup B_2$ and similarly $A_2 \subset A_1 \cup B_1$. Hence we have $\mu(A_1) \leq \mu(A_2) + \mu(B_2) = \mu(A_2)$ and $\mu(A_2) \leq \mu(A_1) + \mu(B_1) = \mu(A_1)$, or $\mu(A_1) = \mu(A_2)$. Thus $\widehat{\mu}$ is well-defined. That it extends μ is trivial. To show completeness, let $D \subset$ (some $B) \in \widehat{\mathcal{A}}_\mu$ with $\mu(B) = 0$. Then $D = \emptyset \cup D \in \widehat{\mathcal{A}}_2 = \widehat{\mathcal{A}}_\mu$.

4. PfS, Exercise 1.2.3, page 16.
Suppose that μ on a field $\mathcal{C}$ is $\sigma-$finite on $\mathcal{C}$ and is extended to $\mathcal{A} = \sigma[\mathcal{C}]$; call the extension μ.
(a) For each $A \in \mathcal{A}$ with $\mu(A) < \infty$ and each $\epsilon > 0$ there exists a set $C = C_\epsilon \in \mathcal{C}$ such that $\mu(A \triangle C) < \epsilon$.
(b) Let μ denote counting measure on the integers. Then $\mathcal{C} = \{C : C$ or C^c is finite$\}$ is a field. Determine $\sigma[\mathcal{C}]$. Show that the conclusion of part (a) fails for the set of even integers.

Solution: Proof of (a): Now

$$\mu(A) = \inf\{\sum_n \mu(A_n) : A \subset \cup_1^\infty A_n \ \text{ with all } \ A_n \in \mathcal{C}\}.$$

Hence there exists $\{A_n\} \subset \mathcal{C}$ such that

$$\sum_{n=1}^{\infty} \mu(A_n) \leq \mu(A) + \epsilon/2\,.$$

Without loss of generality, we may assume that the sets A_n are disjoint (if not, form the disjoint sets $B_1 = A_1$, $B_n = A_1^c \cap \ldots \cap A_{n-1}^c \cap A_n$, $n = 2, 3, \ldots$). Furthermore, there exists an $N = N_\epsilon$ sufficiently large such that $\sum_{N+1}^{\infty} \mu(A_n) \le \epsilon/2$, and hence

$$\sum_{n=1}^{\infty} \mu(A_n) < \sum_{n=1}^{N} \mu(A_n) + \epsilon/2 = \mu(\sum_{n=1}^{N} A_n) + \epsilon/2\,.$$

Then $C \equiv \sum_{n=1}^{N} A_n \in \mathcal{C}$,

$$\mu(A \setminus C) \le \mu(\sum_{n=1}^{\infty} A_n \setminus C) = \mu(\sum_{n=1}^{\infty} A_n) - \mu(C) < \epsilon/2$$

by the choice of N, and

$$\mu(C \setminus A) \le \mu(\sum_{n=1}^{\infty} A_n \setminus A) = \mu(\sum_{n=1}^{\infty} A_n) - \mu(A) < \epsilon/2$$

by the choice of $\{A_n\}$. Putting these together gives

$$\mu(A \bigtriangleup C) = \mu(A \setminus C) + \mu(C \setminus A) < \epsilon/2 + \epsilon/2 = \epsilon.$$

(b) Note that all the singletons $D_k = \{k\}$ are in $\mathcal{C}$, and since all subsets of $\mathbb{Z}$ are either finite or countable, every subset A of $\mathbb{Z}$ can be written as a countable union of the singletons D_k, $k \in A$. Thus $\sigma[\mathcal{C}] = 2^{\mathbb{Z}}$. Consider $A = \{2, 4, 6, \ldots\} = \cup_k \{2k\} \equiv \cup_k C_k$ where each $C_k \in \mathcal{C}$ since C_k itself is a finite set. Note that $A \notin \mathcal{C}$ since neither A nor $A^c = \{1, 3, \ldots\}$ is finite. Thus $\mu(A) = \infty$ (so the hypothesis of (a) fails). Furthermore, for any set $C \in \mathcal{C}$

$$A \Delta C = (A \cap C^c) \cup (A^c \cap C)$$

where both A and A^c are non-finite sets, and either C or C^c is non-finite, and hence at least one of $A \cap C^c$ and $A^c \cap C$ is also non-finite. Thus $\mu(A \Delta C) = \infty$ for all $C \in \mathcal{C}$. Hence the conclusion of (a) fails to hold.

5. PfS, Exercise 1.2.4, page 16. (Nonmeasurable sets). Let Ω consist of the 16 values $1, \ldots, 16$. (Think of them arranged in four rows of four values.) Let

$$C_1 = \{1, 2, 3, 4, 5, 6, 7, 8\}, \qquad C_2 = \{9, 10, 11, 12, 13, 14, 15, 16\},$$
$$C_3 = \{1, 2, 5, 6, 9, 10, 13, 14\}, \quad C_4 = \{3, 4, 7, 8, 11, 12, 15, 16\}.$$

Let $\mathcal{C}$ denote the field generated by $\{C_1, C_2, C_3, C_4\}$, and let $\mathcal{A} = \sigma[\mathcal{C}]$.
(a) Show that $\mathcal{A} \equiv \sigma[\mathcal{C}] \neq 2^\Omega$. (Note that 2^Ω contains $2^{16} = 65,536$ sets.)
(b) Let $\mu(C_i) = 1/2$, $1 \le i \le 4$, with $\mu(C_1C_3) = 1/4$. Show $\hat{\mathcal{A}}_\mu = \mathcal{A}$ with $2^4 = 16$ sets.
(c) Let $\mu(C_i) = 1/2$ for $i = 2, 3, 4$, with $\mu(C_2C_4) = 0$. Show that $\hat{\mathcal{A}}_\mu$ has $2^{10} = 1024$ sets.
(d) Illustrate Proposition 2.1, PfS page 16 in the context of this exercise.

(a) Show that $\mathcal{A} \equiv \sigma[\mathcal{C}] \neq 2^\Omega$.
Proof. Write $\{1, \ldots, 16\}$ in four rows of four numbers each as follows:

	C_3		C_4	
C_1	1	2	3	4
	5	6	7	8
C_2	9	10	11	12
	13	14	15	16

Let $\mathcal{B} \equiv \{C_i \cap C_j : \ i, j \in \{1, \ldots, 4\}\}$. Then it is clear that $\#(\sigma[\mathcal{C}]) = 2^{\mathcal{B}} = 2^4 \neq 2^{16} = \#(2^\Omega)$. Thus $\sigma[\mathcal{C}] \neq 2^\Omega$.

(b) Let $\mu(C_i) = 1/2$ where $1 \le i \le 4$ with $\mu(C_1C_3) = 1/4$. Show that $\hat{\mathcal{A}}_\mu = \mathcal{A}$ with 2^4 sets.
Proof. Let $B_1 = C_1C_3$, $B_2 = C_1C_4$, $B_3 = C_2C_3$, and $B_4 = C_2C_4$. Then $\mu(B_1) = \mu(C_1C_3) = 1/4$ implies that $\mu(B_i) = 1/4$ for $i = 2, 3, 4$. This holds since $1/2 = \mu(C_1) = \mu(B_1 + B_2) = \mu(B_1) + \mu(B_2) = 1/4 + \mu(B_2)$, so that $\mu(B_2) = 1/4$, and similarly $1/2 = \mu(C_3) = \mu(B_1 + B_3) = \mu(B_1) + \mu(B_3) = 1/4 + \mu(B_3)$, so $\mu(B_3) = 1/4$, and $1/2 = \mu(C_4) = \mu(B_2 + B_4) = \mu(B_2) + \mu(B_4) = 1/4 + \mu(B_4)$, so $\mu(B_4) = 1/4$. Thus the only set $B \in \mathcal{A}$ with $\mu(B) = 0$ is $B = \emptyset$, and it follows that $\hat{\mathcal{A}}_\mu = \mathcal{A}$ with $\#(\hat{\mathcal{A}}_\mu) = \#(\mathcal{A}) = 2^4$.

(c) Now suppose $\mu(C_i) = 1/2$ for $i = 2, 3, 4$, but $\mu(C_2C_4) = 0$. Show that $\hat{\mathcal{A}}_\mu$ contains $2^{10} = 1024$ sets.
Proof. In this case $\mu(B_4) = \mu(C_2C_4) = 0$, and this implies that $\mu(B_2) = 1/2 = \mu(B_3)$ (since $\mu(C_2) = 1/2 = \mu(C_4)$. Thus we also have $\mu(B_1) = 0$ (since $\mu(B_1) + \mu(B_2) = 1/2$). Therefore we need to consider all the sets $N \subset 2^{B_1} + 2^{B_4}$ in forming the completion; that is we need to consider

all the sets $\{\{1\}, \{2\}, \{5\}, \{6\}, \{11\}, \{12\}, \{14\}, \{16\}\}$ in forming the completion together with the two basic sets with non-zero probability, B_2 and B_4. Thus $\#(\hat{\mathcal{A}}_\mu) = 2^{10} = 1024$.

(d) Illustrate Proposition 1.2.1 in the context of this exercise.

Proof. Consider μ as given in part (b), and let $B = \{1\}$. Consider extending μ to $\sigma[\hat{\mathcal{A}}_\mu \cup \{B\}]$ by defining $\mu(B) = a$ where $0 < a < 1/4$. This is valid extension of μ for each $a \in (0, 1/4)$, but it is not unique since there are (uncountably) many choices for a.