Statistics 521, Problem Set 10 Solutions

Wellner; 12/04/2019

Reminder: Final exam, 2:30 - 4:20, Wednesday, December 11.

1. Let $X_1, X_2, \ldots$ be independent with $P(X_n = 1) = p_n$ and $P(X_n = 0) = 1 - p_n$. Show that: (i) $X_n \rightarrow_p 0$ if and only if $p_n \rightarrow 0$, and $X_n \rightarrow_{a.s.} 0$ if and only if $\sum_n p_n < \infty$.

   Solution: Here, for any $0 < \epsilon < 1$, $P(|X_n| > \epsilon) = P(X_n = 1) = p_n$. Thus $X_n \rightarrow_p 0$ if and only if $p_n \rightarrow 0$.
   By the first and second Borel-Cantelli lemmas and the first part, for $0 < \epsilon < 1$ we have

   $$P(|X_n| > \epsilon \text{ i.o.}) = \left\{ \begin{array}{c} 0 \\ 1 \end{array} \right\} \quad \text{according as} \quad \sum_{n=1}^{\infty} p_n \left\{ \begin{array}{c} < \infty \\ = \infty \end{array} \right\} .$$

   It follows that $X_n \rightarrow_{a.s.} 0$ if and only if $\sum_{n=1}^{\infty} p_n < \infty$.

2. Exercise 8.4.13, PfS page 168: If $X_1, X_2, \ldots$ are i.i.d. Exponential(1) , then $\limsup_n X_n / \log n = 1$ a.s. and $X_{n:n} / \log n \rightarrow 1$ a.s.

   Solution: (i) Now $P(X_n > x) = e^{-x}$ for all $x > 0$ and $n \geq 1$. Thus with $\epsilon > 0$ we have

   $$P(X_n / \log n \geq (1 \pm \epsilon)) = P(X_n > (1 \pm \epsilon) \log n) = n^{-(1 \pm \epsilon)}.$$

   If follows that

   $$\sum_{n=1}^{\infty} P(X_n / \log n \geq 1 \pm \epsilon) \quad = \quad \sum_{n=1}^{\infty} n^{-(1 \pm \epsilon)} \left\{ \begin{array}{ll} < \infty & \text{for } 1 + \epsilon \\ = \infty & \text{for } 1 - \epsilon. \end{array} \right.$$

   Thus by the first Borel-Cantelli lemma we have

   $$\limsup_n (X_n / \log n) \leq 1 \quad \text{a.s.}$$

   and by the second Borel-Cantelli lemma $\limsup_n (X_n / \log n) \geq 1$ a.s. By combining these two results we see that $\limsup_n (X_n / \log n) = 1$

almost surely.

(ii) Now consider $X_{n:n} \equiv \max_{1 \le k \le n} X_k$. First note that since $X_n \le \max_{1 \le k \le n} X_k = X_{n:n}$ we have

$$1 = \limsup_n \frac{X_n}{\log n} \le \limsup_n \frac{X_{n:n}}{\log n} \quad a.s. \tag{1}$$

To show that

$$\limsup_n \frac{X_{n:n}}{\log n} \le 1, \tag{2}$$

a.s., let $\epsilon > 0$. Since $\limsup_n (X_n / \log n) \le 1$ a.s., for all ω in a set with probability 1 we have $X_k / \log k \le 1 + \epsilon$ for all $k \ge$ some N_ω. But then

$$\begin{aligned} \frac{X_{n:n}}{\log n} &= \max_{1 \le k \le n} \frac{X_k}{\log n} \\ &\le \max \left\{ \max_{1 \le k \le N_\omega} \frac{X_k}{\log n} \; , \; \max_{N_\omega \le k \le n} \frac{X_k}{\log k} \right\} \\ &\le \max \left\{ \frac{(1+\epsilon) \log N_\omega}{\log n} \; , \; (1+\epsilon) \right\} \\ &\le \epsilon + (1+\epsilon) = 1 + 2\epsilon. \end{aligned}$$

for $n \ge \widetilde{N}_\omega$ where $\widetilde{N}_\omega$ is so large that $(1+\epsilon)\log N_\omega / \log n < \epsilon$ for all $n \ge \widetilde{N}_\omega$. Since $\epsilon > 0$ is arbitrary, this shows that (2) holds. Combining (1) with (2) yields the claimed limit for $\limsup_n (X_{n:n}/\log n)$.

To show that $X_{n:n}/\log n \to_{a.s.} 1$, it now suffices to show that

$$\liminf_n X_{n:n}/\log n \ge 1 \quad \text{a.s.},$$

or, equivalently, to show that $P(X_{n:n}/\log n \le (1-\epsilon) \text{ i.o.}) = 0$ for every $\epsilon \in (0,1)$. But since the X_k's are i.i.d. exponential(1)

$$\begin{aligned} P(X_{n:n} \le (1-\epsilon)\log n) &= P(\cap_{k=1}^n [X_k \le (1-\epsilon)\log n]) = P(X_1 \le (1-\epsilon)\log n)^n \\ &= (1 - \exp(-(1-\epsilon)\log n))^n = \left(1 - \frac{1}{n^{1-\epsilon}}\right)^n \\ &= \left(1 - \frac{n^\epsilon}{n}\right)^n \le e^{-n^\epsilon} \end{aligned}$$

since $1 - y \le e^{-y}$ so that $(1 - x/n)^n \le (e^{-x/n})^n = e^{-x}$. Thus we find that $\sum_1^\infty P(X_{n:n} \le (1-\epsilon)\log n) \le \sum_1^\infty e^{-n^\epsilon} < \infty$ and hence $P(X_{n:n} \le (1-\epsilon)\log n \text{ i.o.}) = 0$. We conclude that $X_{n:n}/\log n \to_{a.s.} 1$.

3. Exercise 8.4.14, PfS page 168: If $X_1, X_2, \ldots$ are i.i.d. Normal$(0,1)$, then $X_{n:n}/\sqrt{2\log n} \to_p 1$. Hint: You may use the following inequalities for the tail probability of a standard normal distribution:

$$\frac{z}{1+z^2}\phi(z) \le P(Z \ge z) < \frac{1}{z}\phi(z)$$

where $\phi(z) = (2\pi)^{-1/2}e^{-z^2/2}$.

Solution: My (overly ambitious) statement of this problem replaced with $\to_p$ replaced by $\to_{a.s.}$. Thanks to to several of you for pointing out the resulting difficulties. The complete solution given below for the a.s. statement given is due to Omid Sadeghi and Zhenman Yuan.

To prove that that $\limsup_n X_{n:n}/\sqrt{2\log n} \le 1$ almost surely we will first show that $\limsup_n X_n/\sqrt{2\log n} = 1$ a.s. To see this, note that by the hint (or, slightly more simply, by the upper bound version of the inequality given in the bonus problem # 5),

$$P(X_n > \sqrt{(1+\epsilon)2\log n}) \le 2^{-1}\exp(-(1+\epsilon)\log n) \le n^{-(1+\epsilon)},$$

and hence $\sum_{n=1}^{\infty} P(X_n > \sqrt{(1+\epsilon)2\log n}) < \infty$. By the Borel-Cantelli lemma this implies that $P(X_n > \sqrt{(1+\epsilon)2\log n}$ i.o.$) = 0$. This implies that

$$\limsup_n \frac{X_n}{\sqrt{2\log n}} \le 1 \quad a.s.$$

Similarly, from the lower bound in the hint we find that

$$\begin{aligned} P(X_n > \sqrt{2\log n}) &\ge \frac{\sqrt{2\log n}}{1+2\log n}n^{-1} \\ &\sim \frac{1}{\sqrt{2\log n}}n^{-1} \end{aligned}$$

and hence for some N_0

$$\sum_{n=N_0}^{\infty} P(X_n > \sqrt{2\log n}) \ge 2^{-1}\sum_{n\ge N_0} \frac{\sqrt{2\log n}}{1+2\log n}n^{-1} = \infty.$$

By the second Borel-Cantelli lemma we conclude that $P(X_n > \sqrt{2\log n}$ i.o.$) = 1$. Thus $\limsup_n \frac{X_n}{\sqrt{2\log n}} \ge 1$ almost surely. Combining the two results yields

$$\limsup_n \frac{X_n}{\sqrt{2\log n}} = 1 \quad \text{almost surely.}$$

Now consider $X_{n:n} = \max_{1\le k\le n} X_k$. As in the previous problem, $X_n \le X_{n:n}$ and hence

$$1 = \limsup_n \frac{X_n}{\sqrt{2\log n}} \le \limsup_n \frac{X_{n:n}}{\sqrt{2\log n}}, \quad \text{a.s.}$$

so we know that $1 \le \limsup_n \frac{X_{n:n}}{\sqrt{2\log n}}$ almost surely. On the other hand, Since $\limsup_n (X_n/\sqrt{2\log n}) \le 1$ a.s., for all ω in a set with probability 1 we have $X_k/(2\log k)^{1/2} \le 1+\epsilon$ for all $k \ge$ some N_ω. But then

$$\begin{aligned}
\frac{X_{n:n}}{\sqrt{2\log n}} &= \max_{1\le k\le n} \frac{X_k}{\sqrt{2\log n}} \\
&\le \max\left\{ \max_{1\le k\le N_\omega} \frac{X_k}{\sqrt{2\log n}} \, , \, \max_{N_\omega\le k\le n} \frac{X_k}{\sqrt{2\log k}} \right\} \\
&\le \max\left\{ \frac{(1+\epsilon)\sqrt{2\log N_\omega}}{\sqrt{2\log n}} \, , \, (1+\epsilon) \right\} \\
&\le \epsilon + (1+\epsilon) = 1+2\epsilon.
\end{aligned}$$

for $n \ge \widetilde{N}_\omega$ where $\widetilde{N}_\omega$ is so large that $(1+\epsilon)\log N_\omega/\sqrt{2\log n} < \epsilon$ for all $n \ge \widetilde{N}_\omega$. Since $\epsilon > 0$ is arbitrary, we conclude that $\limsup_n \frac{X_{n:n}}{\sqrt{2\log n}} \le 1$ almost surely. Combining the upper and lower bounds yields

$$\limsup_n \frac{X_{n:n}}{\sqrt{2\log n}} = 1 \quad \text{almost surely.} \tag{3}$$

To prove that $\lim_{n\to\infty}(X_{n:n}/\sqrt{2\log n}) = 1$ a.s. it now suffices to show that

$$1 \le \liminf_n \frac{X_{n:n}}{\sqrt{2\log n}} \quad \text{a.s..} \tag{4}$$

This will follow if we show that

$$P(\max_{1\le k\le n} X_k < \sqrt{2(1-\epsilon)\log n} \;\; \text{i.o.}) = 0. \tag{5}$$

But by the lower bound of the hint,

$$
\begin{aligned}
P(\max_{k\le n} X_k < \sqrt{2(1-\epsilon)\log n}) &= P(X_1 < \sqrt{2(1-\epsilon)\log n})^n \\
&= \left(1 - P(X_1 \ge \sqrt{2(1-\epsilon)\log n}\right)^n \\
&\le \left(1 - \frac{1}{\sqrt{2\pi}} \frac{\sqrt{2(1-\epsilon)\log n}}{1+2(1-\epsilon)\log n} n^{-(1-\epsilon)}\right)^n \\
&\equiv \left(1 - \frac{c_n n^\epsilon}{n}\right)^n \\
&\le \exp(-c_n n^\epsilon) \le e^{-n^{\epsilon/2}}
\end{aligned}
$$

for $n \ge$ some N_ϵ since $c_n n^{\epsilon/2} \sim (2\pi)^{-1/2}(\log n)^{-1/2} n^{\epsilon/2} \to \infty$ as $n \to \infty$. Therefore $\sum_{n=1}^\infty \exp(-n^{\epsilon/2}) < \infty$, it follows by the Borel-Cantelli lemma that (5) holds. Hence (4) holds. Combined with (3) we conclude that $X_{n:n}$ satisfies

$$\frac{X_{n:n}}{\sqrt{2\log n}} \to_{a.s.} 1.$$

Remark: This problem is connected to a result known as *Sudakov's lower bound*: in fact it holds that

$$\lim_{n\to\infty} \frac{E(X_{n:n})}{\sqrt{2\log n}} = 1$$

and

$$E(X_{n:n}) \ge (\pi \log 2)^{-1/2} \sqrt{\log n}$$

for all $n \ge 1$. See e.g. Giné and Nickl, *Mathematical Foundations of Infinite-Dimensional Statistical Models*, page 56.

4. (Monte Carlo Estimation) Exercise 8.5.2, PfS page 174: Let $h : [0,1] \to [0,1]$ be continuous.
(i) Let $X_k \equiv 1_{[h(\xi_k) \ge \Theta_k]}$, where $\xi_1, \xi_1, \ldots$ and $\Theta_1, \Theta_2, \ldots,$ are i.i.d Uniform$[0,1]$ random variables. Show that $\overline{X}_n \to_{a.s.} \int_0^1 h(t)dt$.
(ii) Let $Y_k \equiv h(\xi_k)$. Show that $\overline{Y}_n \to_{a.s.} \int_0^1 h(t)dt$.
(iii) Evaluate $Var(\overline{X}_n)$ and $Var(\overline{Y}_n)$ and compare them.

Solution: (i) Note that $X_1, X_2, \ldots$ are i.i.d. with $E|X_1| < \infty$ since $\|h\|_\infty \le 1$. Furthermore

$$E(X_1) = E1_{[h(\xi_1)\ge\Theta_1]} = \int_0^1 \int_0^1 1_{[h(t)\ge s]} ds dt = \int_0^1 h(t)dt$$

By the SLLN we conclude that $\overline{X}_n \to_{a.s.} E(X_1) = \int_0^1 h(t)dt$.
(ii) Also, $Y_1, Y_2, \ldots$ are i.i.d. with $E|Y_1| < \infty$ since h is bounded. Furthermore $E(Y_1) = Eh(\xi_1) = \int_0^1 h(t)dt$. Thus by the SLLN it follows that $\overline{Y}_n \to_{a.s.} E(Y_1) = \int_0^1 h(t)dt$.
(iii) Now

$$\begin{aligned} Var(\overline{X}_n) &= n^{-1}Var(X_1) = n^{-1}\{E(X_1^2) - (E(X_1))^2\}, \\ Var(\overline{Y}_n) &= n^{-1}Var(Y_1) = n^{-1}\{E(Y_1^2) - (E(Y_1))^2\}, \end{aligned}$$

where

$$E(X_1^2) = E1^2_{[h(\xi_1)\ge\Theta_1]} = E1_{[h(\xi_1)\ge\Theta_1]} = \int_0^1 h(t)dt$$

as in (i), and

$$E(Y_1)^2 = Eh^2(\xi_1) = \int_0^1 h^2(t)dt.$$

Thus

$$\begin{aligned} \frac{Var(\overline{Y}_n)}{Var(\overline{X}_n)} &= \frac{Var(h(\xi_1)}{Var(1_{[h(\xi_1)\ge\Theta_1]})} \\ &= \frac{\int_0^1 h^2(t)dt - (\int_0^1 h(t)dt)^2}{\int_0^1 h(t)dt\left(1 - \int_0^1 h(t)dt\right)} \\ &\le \frac{\|h\|_\infty \int_0^1 h(t)dt - (\int_0^1 h(t)dt)^2}{\int_0^1 h(t)dt\left(1 - \int_0^1 h(t)dt\right)} \\ &\le 1 \end{aligned}$$

where the last inequality follows from $\|h\|_\infty \le 1$.

For example, if $h(t) = 1_{[0,1/2]}(t)$, then $\int_0^1 h(t)dt = 1/2 = \int_0^1 h^2(t)dt$, and

$$\frac{Var(\overline{Y}_n)}{Var(\overline{X}_n)} = 1.$$

If $h(t) = t$ for $t \in [0,1]$, then $Var(h(\xi_1)) = Var(\xi_1) = 1/12$, $\int_0^1 h(t)dt = 1/2$, so

$$\frac{Var(\overline{Y}_n)}{Var(\overline{X}_n)} = \frac{1/12}{(1/2)(1-1/2)} = \frac{1}{3}.$$

This is not too surprising since the Monte-Carlo scheme based on the Y_k's involves less randomness than the scheme based on the X_k's.

5. **Bonus problem:** Show that $P(Z \geq z) \leq (1/2)e^{-z^2/2}$. When is this bound better than the upper bound in the hint for problem 3?

   Solution: First consider the function

$$g(z) \equiv (1-\Phi(z))e^{z^2/2} = \frac{1-\Phi(z)}{e^{-z^2/2}}.$$

   Note that $g(0) = (1/2)\cdot 1 = 1/2$. Furthermore

$$\begin{aligned} g'(z) &= -\phi(z)e^{z^2/2} + (1-\Phi(z))e^{z^2/2}\cdot z \\ &= e^{z^2/2}\{z(1-\Phi(z)) - \phi(z)\} \\ &\leq e^{z^2/2}\cdot 0 = 0 \end{aligned}$$

   since $1-\Phi(z) \leq z^{-1}\phi(z)$. It follows that $g(z) \leq 1/2$ for all $z \geq 0$. Or, equivalently, $P(Z \geq z) = 1-\Phi(z) \leq (1/2)e^{-z^2/2}$ for all $z \geq 0$.

   Now consider the function $h(z) \equiv (1/2)e^{-z^2/2} - (1-\Phi(z))$. Note that $h(0) = 0$. On the other hand we now have

$$\begin{aligned} h'(z) &= -\frac{z}{2}e^{-z^2/2} + \phi(z) \\ &= \left\{\frac{1}{\sqrt{2\pi}} - \frac{z}{2}\right\} e^{-z^2/2} \\ &\geq 0 \end{aligned}$$

   for $z \leq \sqrt{2/\pi}$. Dividing through the last inequality by $z > 0$ we see that this is exactly where $2^{-1}e^{-z^2/2}$ is smaller than $z^{-1}\phi(z)$. See Figure 1.

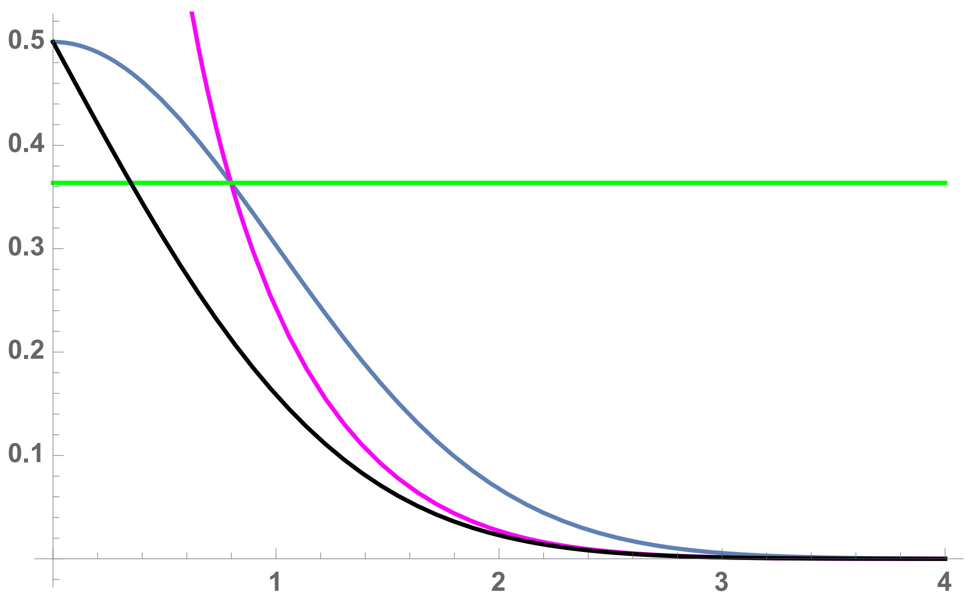

Figure 1: Gaussian tail bounds: $2^{-1}e^{-z^2/2}$ (blue); $z^{-1}\phi(z)$ (magenta); $1 - \Phi(z) = P(Z > z)$ (black); $z^{-1}\phi(\sqrt{2/\pi}) = 2^{-1}e^{-z^2/2}$ (green).