Statistics 521, Problem Set 1 Solutions

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1. (a) Suppose that $\{\mathcal{A}_n\}$ is an increasing sequence of algebras, i.e. $\mathcal{A}_n \subset \mathcal{A}_{n+1}$ for all $n \geq 1$. Show that $\cup_{n=1}^{\infty} \mathcal{A}_n$ is an algebra.
(b) Suppose that the $\mathcal{A}_n$ of (a) are $\sigma-$algebras. Show by constructing a counter-example that $\cup_{n=1}^{\infty} \mathcal{A}_n$ need not be a $\sigma-$algebra.

Solution: (a) If $A \in \cup_{n=1}^{\infty} \mathcal{A}_n$, then $A \in \mathcal{A}_m$ for some m, and since $\mathcal{A}_m$ is an algebra, $A^c \in \mathcal{A}_m$. Hence $A^c \in \cup_{n=1}^{\infty} \mathcal{A}_n$. If $A, B \in \cup_{n=1}^{\infty} \mathcal{A}_n$, then $A \in \mathcal{A}_m$ for some m and $B \in \mathcal{A}_n$ for some n. Without loss we can assume that $m \leq n$, and since $\mathcal{A}_m \subset \mathcal{A}_n$ it follows that $A, B \in \mathcal{A}_n$. Since $\mathcal{A}_n$ is an algebra, it follows that $A \cup B \in \mathcal{A}_n$, and hence that $A \cup B \in \cup_{n=1}^{\infty} \mathcal{A}_n$.
(b) Take $\Omega = [0,1]$. Let $\mathcal{A}_1 = \{\emptyset, \Omega\}$, $\mathcal{A}_2 = \sigma[\mathcal{A}_0, [0, 1/2)]$, ..., $\mathcal{A}_n = \sigma[\mathcal{A}_{n-1}, [0, 1-1/n)]$, ... Then $\mathcal{A}_n \subset \mathcal{A}_{n+1}$ by construction, but $\cup_{n=1}^{\infty} \mathcal{A}_n$ is not a sigma field: if we let $A_k = [0, 1-1/k)$ for each $k = 1, 2, \ldots$, then $A_k \in \cup_{n=1}^{\infty} \mathcal{A}_n$ since $A_k \in \mathcal{A}_k$ by construction, but $[0,1) = \cup_{k=1}^{\infty} A_k \notin \cup_{n=1}^{\infty} \mathcal{A}_n$.

2. Proposition 1.1(b), PfS, page 3: There exists a minimal field, $\sigma-$field, or monotone class generated by (or containing) any specified class $\mathcal{C}$ of subsets of Ω.

Solution: By proposition 1.1.1(i), arbitrary intersections of fields, $\sigma-$fields, or monotone classes are again fields, $\sigma-$fields, or monotone classes. Hence

$$\phi[\mathcal{C}] \equiv \bigcap \{\mathcal{A}_\alpha : \mathcal{A}_\alpha \text{ is a -field of subsets of } \Omega \text{ for which } \mathcal{C} \subset \mathcal{A}_\alpha\}$$

is again a field, and it is the smallest such field: if $\mathcal{D}$ is the minimal field containing $\mathcal{C}$ so that $\mathcal{D} \subset \phi[\mathcal{C}]$, then we also have $\phi[\mathcal{C}] \subset \mathcal{D}$ by construction of $\phi[\mathcal{C}]$, and hence $\phi[\mathcal{C}] = \mathcal{D}$. The argument is the same for $\sigma-$fields and monotone classes with $\phi[\mathcal{C}]$ replaced by $\sigma[\mathcal{C}]$ and $\text{mon}[\mathcal{C}]$ respectively.

3. PfS, Exercise 1.1.1, PfS, page 4: Let $\mathcal{C}_1$ and $\mathcal{C}_2$ denote two collections of subsets of the set Ω. If $\mathcal{C}_1 \subset \sigma[\mathcal{C}_2]$ and $\mathcal{C}_2 \subset \sigma[\mathcal{C}_1]$, then $\sigma[\mathcal{C}_1] = \sigma[\mathcal{C}_2]$.

   Solution: Since $\mathcal{C}_1 \subset \sigma[\mathcal{C}_2]$, it follows immediately that $\sigma[\mathcal{C}_1] \subset \sigma[\sigma[\mathcal{C}_2]] = \sigma[\mathcal{C}_2]$. By a symmetric argument $\sigma[\mathcal{C}_2] \subset \sigma[\mathcal{C}_1]$. Hence $\sigma[\mathcal{C}_2] = \sigma[\mathcal{C}_1]$.

4. PfS, Exercise 1.1.2, PfS, page 8. We always have $\mu(\liminf A_n) \le \liminf \mu(A_n)$, while $\limsup \mu(A_n) \le \mu(\limsup A_n)$ if $\mu(\Omega) < \infty$.

   Solution: This problem was carried over to problem set #2.

5. PfS(2000), Exercise 9.1.4, page 182; or PfS(2012), Exercise A.1.4, page 428. Suppose that $X_n \sim \text{Binomial}(n, p_n)$ where $np_n \to \lambda > 0$. Show that

$$P(X_n = k) \to \frac{\lambda^k}{k!} \exp(-\lambda) = P(Y = k)$$

   where $Y \sim \text{Poisson}(\lambda)$; this implies that $X_n \to_d Y$. Can this be strengthened?

   Solution:

$$\begin{aligned} P(X_n = k) &= \binom{n}{k} p_n^k (1 - p_n)^{n-k} \\ &= \frac{n(n-1)\cdots(n-k+1)}{k! n^k} (np_n)^k (1 - \frac{np_n}{n})^{n-k} \\ &\to \frac{1}{k!} \lambda^k e^{-\lambda} \end{aligned}$$

   since $(1 + x_n/n)^n \to e^x$ if $x_n \to x$. This can be strengthened in several ways as we will see later. In particular, with $Q_n(A) = P(X_n \in A)$ and $P_\lambda(Y \in A)$ for any $A \in 2^{\mathbb{N}}$, this pointwise convergence (for each fixed k) implies that

$$\begin{aligned} d_{TV}(Q_n, P_\lambda) &\equiv \sup_{A \in 2^{\mathbb{N}}} |Q_n(A) - P_\lambda(A)| \\ &= \frac{1}{2} \sum_{k=0}^{\infty} |Q_n(\{k\}) - P_\lambda(\{k\})| \to 0; \end{aligned}$$

this follows from Scheffé's theorem. In fact it is known that

$$d_{TV}(Q_n, P_n) \le (1 \vee np_n)^{-1} \frac{(np_n)^2}{n}.$$

See Durrett pages 101-102 and 146 - 149 for a slightly weaker inequality via a coupling argument due to Hodges and Le Cam (1960). Also see Exercise 5.4, PfS page 294 for an extension to the case when X_n is replaced by $S_n \equiv \sum_{k=1}^n X_{n,k}$ for independent (but not necessarily identically distributed) Bernoulli($p_{n,k}$) random variables.