Statistics 521, Final Exam, Solutions

Wellner; Wednesday, 12/12/2019

1. (36 points). **Define** four of the following eight terms:
   (a) *Almost sure convergence* of a sequence of random variables $\{X_n\}$.
   (b) A *uniformly integrable* sequence of random variables $\{X_n\}$.
   (c) A median m_F of a distribution function F and a particular median in terms of F^{-1}.
   (d) A *symmetric random variable* X.
   (e) The *tail* $\sigma-$*field* of a sequence of random variables $X_1, X_2, \ldots$.
   (f) *Khintchine - equivalent* sequences of random variables.
   (g) *Absolute continuity* of a signed measure ϕ with respect to a measure μ, **and** *singularity* of ϕ with respect to μ.
   (h) The *product* $\sigma-$*field* $\mathcal{A} \times \mathcal{A}'$ for two measurable spaces $(\Omega, \mathcal{A})$ and $(\Omega', \mathcal{A}')$.

   Solution: See PfS, Chapters 1-8.

2. (40 points). Give careful **statements** of four of the following eight theorems or results:
   (a) Fatou's lemma.
   (b) The dominated convergence theorem.
   (c) The Helly - Bray theorem.
   (d) The Mann-Wald theorem.
   (e) The Kolmogorov zero-one law.
   (f) The second Borel-Cantelli lemma.
   (g) Kolmogorov's maximal inequality
   (h) Lévy's maximal inequality.

   Solution: See PfS, Chapters 1-8.

3. (40 points) Let $(\Omega, \mathcal{A}, \mu)$ be a fixed measure space, and let X denote a measurable function from $(\Omega, \mathcal{A}, \mu)$ to $(\overline{\mathbb{R}}, \overline{\mathcal{B}})$.
   (a) Define a *simple function* X on $(\Omega, \mathcal{A})$.
   (b) Define the *Lebesgue integral* $\int X d\mu$ for a simple function X as in (a).
   (c) Now let $X \geq 0$ be a non-negative measurable function. Define the *Lebesgue integral* $\int X d\mu$ for such a function X.
   (d) For $X \geq 0$ as in (c) give a particular sequence $\{X_n\}$ of simple functions such that $X_n \geq 0$ and $X_n \nearrow X$.
   (e) Now define the *Lebesgue integral* $\int X d\mu$ for a general measurable function X.

   Solution: See PfS, Sections 2.2 and 3.1.

4. (42 points). Let X, X_1, $X_2, \ldots$ be i.i.d. with d.f. $F(x) = 1 - x^{-\alpha}$ for $x \geq 1$ where $\alpha > 0$.
   (a) For what values of $r > 0$ do we have $E(X^r) < \infty$? For what values of r do we have $x^r P(X > x) \to 0$?
   (b) Find a sequence b_n so that $\limsup_{n\to\infty}(\log X_n / b_n) = 1$ almost surely.
   (c) Let $M_n \equiv \max_{1\leq k\leq n} X_k$. Find the distribution function F_n of M_n. Find the density function f_n of M_n.
   (d) Let $M_n \equiv \max_{1\leq k\leq n} X_k$ as in (c). Use (c) to find a sequence of numbers c_n so that if $\widetilde{F}_n$ is the distribution function of M_n/c_n, then $M_n/c_n \to_d$ "something" and find the distribution function of "something".
   (e) Find the density functions $\widetilde{f}_n$ corresponding to the distribution functions $\widetilde{F}_n$, and show that $\widetilde{f}_n(x) \to \widetilde{f}(x)$ for each $x \in \mathbb{R}$.
   (f) What can you conclude from the convergence in (e)?

   Solution: (a) For $r > 0$ we have

$$
\begin{aligned}
E(X^r) &= \int_0^\infty r x^{r-1}(1 - F(x))dx = \int_0^1 r x^{r-1} \cdot 1\, dx + \int_1^\infty r x^{r-1} x^{-\alpha} dx \\
&= 1 + r \int_1^\infty x^{-(\alpha - r) - 1} dx = 1 + \frac{r}{\alpha - r} \\
&= \begin{cases} \frac{\alpha}{\alpha - r}, & \text{if } \alpha > r \\ \infty, & \text{if } \alpha \leq r. \end{cases}
\end{aligned}
$$

Furthermore $x^r P(X > x) = x^{-(\alpha - r)} \to 0$ if $\alpha > r$.
(b) Now $P(\log X_n > y) = P(X_n > e^y) = e^{-\alpha y}$ and hence with $y = \alpha^{-1}(1 \pm \epsilon)\log n$,

$$P(\log X_n > (1 \pm \epsilon)\alpha^{-1}\log n) = e^{-(1\pm\epsilon)\log n} = \frac{1}{n^{1\pm\epsilon}}.$$

It follows that

$$\sum_{n=1}^{\infty} P(\log X_n > (1 \pm \epsilon)\alpha^{-1}\log n) \begin{cases} < \infty, & \text{if } \epsilon > 0, \\ = \infty, & \text{if } \epsilon \le 0. \end{cases}$$

By the first and second Borell-Cantelli lemmas we find that

$$\limsup_n (\log X_n)/(\alpha^{-1}\log n) = 1 \quad \text{a.s.}$$

(c) The distribution functions F_n of $M_n \equiv \max_{k \le n} X_k$ are given by

$$\begin{aligned} F_n(x) &= P(\max_{k \le n} X_k \le x) \\ &= P(X_1 \le x)^n = (1 - P(X_1 > x))^n = (1 - x^{-\alpha})^n \quad \text{for } \; x \ge 1, \end{aligned}$$

and the density functions $f_n = F_n'$ are

$$f_n(x) = n\left(1 - x^{-\alpha}\right)^{n-1}(\alpha x^{-\alpha-1}) \;\; \text{for} \;\; x \ge 1.$$

(d) Now

$$\begin{aligned} \widetilde{F}_n(x) &\equiv P(M_n/c_n \le x) = P(M_n \le c_n x) = F_n(c_n x) \\ &= (1 - (c_n x)^{-\alpha})^n \quad \text{if } \; c_n x \ge 1 \\ &= \left(1 - \frac{x^{-\alpha}}{n}\right)^n \quad \text{if } c_n^{-\alpha} = 1/n \\ &\to \exp(-x^{-\alpha}) \equiv \widetilde{F}(x) \end{aligned}$$

as $n \to \infty$. Thus the claimed convergence in distribution holds with $c_n = n^{1/\alpha}$. Note that $\lim_{x\to\infty} \widetilde{F}(x) = e^{-0} = 1$ and $\lim_{x \searrow 0} \widetilde{F}(x) = e^{-\infty} = 0$, and by the density calculation in (e) below $\widetilde{F}$ is monotone increasing. Thus $\widetilde{F}$ is a distribution function.

(e) The density functions of the distribution functions $\widetilde{F}_n$ are given by

$$\begin{aligned} \widetilde{f}_n(w) &= \widetilde{F}_n'(x) \\ &= n\left(1-\frac{x^{-\alpha}}{n}\right)^{n-1}\frac{\alpha}{n}x^{-\alpha-1} \\ &= \alpha x^{-\alpha-1}\left(1-\frac{x^{-\alpha}}{n}\right)^{n-1} \\ &\to \alpha x^{-\alpha-1}\exp(-x^{-\alpha}) \equiv \widetilde{f}(x). \end{aligned}$$

for each fixed x. Since the densities $\widetilde{f}_n$ converge pointwise to the density $\widetilde{f}$, it follows from Scheffé's lemma that

$$d_{TV}(\tilde{P}_n, \tilde{P}) = \frac{1}{2}\int |\widetilde{f}_n(x) - \widetilde{f}(x)|dx \to 0.$$

It would be interesting to know how fast this convergence occurs; that is, for what sequences $r_n \searrow 0$ do we have

$$d_{TV}(\widetilde{P}_n, \widetilde{P}) \le r_n?$$

Do **either** 5 **or** 6:

5. (30 points)
(a) Give an example of a sequence of non-negative random variables X_n, X, all defined on a common probability space $(\Omega, \mathcal{A}, P)$ satisfying $X_n \to_{a.s.} X$, but $E(X_n) \nrightarrow E(X)$.
(b) Suppose that $X_n \equiv (n/\log n)1_{[0,1/n]}(U)$ for $n \ge 3$. Show that $\{X_n\}$ is uniformly integrable and $E(X_n) \to 0$ even though the sequence $\{X_n\}$ is not dominated by any integrable random variable Y.
(c) Give an example of a sequence of non-negative random variables X_n, X, all defined on a common probability space $(\Omega, \mathcal{A}, P)$ satisfying $E(X_n) \to E(X)$, but $X_n \nrightarrow_{a.s.} X$.

Solution:
(a) Let $X_n = n^{\alpha}1_{[0,1/n)}(U)$ where $U \sim \text{Uniform}(0,1)$. Then $X_n \to_{a.s.} 0 \equiv X$ since $U > 0$ a.s. and hence $1/n < U(\omega)$ for $n \ge \text{some}N_\omega$ for all ω in a set with probability 1. On the other hand

$$E(X_n) = n^{\alpha}\cdot n^{-1}\begin{cases} \nearrow \infty & \text{if } \alpha > 1, \\ = 1 & \text{if } \alpha = 1, \\ \searrow 0, & \text{if } \alpha < 1. \end{cases}$$

Thus $E(X_n) \not\to 0 = E(X)$ if $\alpha \ge 1$.
(b) (b) Now $X_n \ge 0$, $X_n \to_p 0 \equiv X$, and $E(X_n) = 1/\log(n) \to 0 = E(X)$ as $n \to \infty$. Thus $\{X_n\}$ is uniformly integrable by Vitali's theorem. However the smallest rv above X_n for all $n \ge 3$ is the rv $Y = \sum_{k=3}^{\infty} \frac{k}{\log k} 1_{(1/(k+1),1/k]}(U)$, and this has expectation

$$
\begin{aligned}
E(Y) &= \sum_{k=3}^{\infty} \frac{k}{\log(k)} \left\{ \frac{1}{k} - \frac{1}{k+1} \right\} \\
&= \sum_{k=3}^{\infty} \frac{k}{\log(k)} \frac{1}{k(k+1)} \\
&= \sum_{k=3}^{\infty} \frac{1}{(k+1)\log(k)} = \infty\,.
\end{aligned}
$$

(c) Let $U \sim \text{Uniform}(0,1)$, and set $Y_{m,k} \equiv 1_{(k-1)/2^m, k/2^m]}(U)$ for $k \in \{1, \ldots, 2^m\}$ and $m \ge 1$. Then $E(X_n) = E(Y_{m,k}) = 2^{-m} \to 0$ as $n = 2(2^{m-1} - 1) + k \to \infty$, but $X_n = Y_{m,k} > 0$ i.o. with probability 1 since $P(U \in (0,1]) = 1$, so $X_n \to_1 0 \equiv X$, but $X_n \not\to_{a.s.} 0$.

6. (30 points). In problem 4 of Problem Set # 9 you showed that if $P(A_n) \to 0$ and $\sum_{n=1}^{\infty} P(A_n \cap A_{n+1}^c) < \infty$ for some events $\{A_n\}$, then $P(A_n \text{ i.o.}) = 0$. To illustrate the utility of this result, find an example of events $\{A_n\}$ with $P(A_n) \to 0$, $\sum_1^{\infty} P(A_n \cap A_{n+1}^c) < \infty$, so that $P(A_n \text{i.o.}) = 0$, by problem 9.4, but $\sum_{n=1}^{\infty} P(A_n) = \infty$.
Hint: Define the events A_n in terms of a random variable U with a uniform distribution on $[0,1]$.

Solution: Let $U \sim \text{Uniform}(0,1)$ and let $A_n \equiv \{U \le 1/n\}$ for $n \ge 1$. Then $P(A_n) = P(U \le 1/n) = 1/n \to 0$ and $\sum_{n=1}^{\infty} P(A_n) = \sum_1^{\infty} n^{-1} = \infty$. But $P(A_n \cap A_n^c) = P(1/(n+1) < U \le 1/n) = 1/n - 1/(n+1) = 1/(n(n+1))$, and hence $\sum_1^{\infty} P(A_n \cap A_n^c) = \sum_1^{\infty} 1/(n(n+1)) = 1 < \infty$, so by problem 4 of Problem set # 9 we have $P(A_n \text{ i.o.}) = 0$.

Do **either** 7 **or** 8:

7. (25 points). Suppose that P is the measure with density $p(x) = (1/2)1_{[0,2]}(x)$ with respect to Lebesgue measure λ (on the Borel $\sigma-$field

of $\mathbb{R}$), and Q is the measure with density $q(x) = (1/3)1_{[1,4]}(x)$ with respect to Lebesgue measure λ.
(a) Is $P \ll Q$? Why or why not?
(b) Give the Lebesgue decomposition of P with respect to Q.
(c) Is $Q \ll P + Q$? Why or why not?
(d) Give the Lebesgue decomposition of Q with respect to $P + Q$.
(e) If $\phi \equiv P - Q$, find $|\phi|(\mathbb{R})$.

Solution: (a) No. Since $Q([0, 1]) = 0$, but $P([0, 1]) = 1/2 > 0$.
(b) The Lebesgue decomposition of P with respect to Q is given by

$$\begin{aligned}
P &= P_{ac} + P_s \qquad \text{where} \\
P_{ac}(A) &= \int_{A\cap[1,2]} (3/2) \cdot dQ = (1/2)\lambda(A \cap [1, 2]), \\
P_s(A) &= \int_{A\cap[0,1)} (1/2)dx = (1/2)\lambda(A \cap [0, 1]).
\end{aligned}$$

Note that $P_s([1, 4]) = 0$ while $P_{ac}([1, 4]^c) = 0$.
(c) Yes. If $(P + Q)(A) = 0$, then both $P(A) = 0$ and $Q(A) = 0$, so in particular $Q \ll P + Q$.
(d) We can write

$$\begin{aligned}
Q(A) &= \int_A dQ = \int_A q d\lambda = (1/3)\lambda(A \cap [1, 4]) \\
&= \int_A \frac{q}{p+q}(p + q)d\lambda = \int_A \frac{q}{p+q} d(P + Q)
\end{aligned}$$

where $q/(p + q) = (1/3)/((1/2) + (1/3)) = 2/5$ on $[1, 2]$, $q/(p + q) = 1$ on $(2, 4]$.
(e) If $\phi = P - Q$ so that $\phi(A) = \int_A d(P - Q) = \int_A (p - q)d\lambda$, where $p - q = (1/2)1_{[0,1]} - (1/3)1_{[1,4]}$, then $\Omega^+ = [0, 2]$ and $\Omega^- = [0, 2]^c$

$$\begin{aligned}
\phi^+(A) &= \int_A (2^{-1}1_{[0,1)}(x) + (1/6)1_{[1,2)})dx, \\
\phi^-(A) &= (1/3)1_{[2,4)}(x)dx,
\end{aligned}$$

and hence

$$\begin{aligned}
|\phi|(\mathbb{R}) &= (1/2)\int_{\mathbb{R}} \{1_{[0,1)} + (1/6)1_{[1,2)}\}d\lambda + (1/3)\int_{\mathbb{R}} 1_{[2,4)}(x)dx \\
&= (1/2) + (1/6) + (2/3) = 4/3.
\end{aligned}$$

Note that $d_{TV}(P, Q) = (1/2) \int |p - q| d\lambda = (1/2)(4/3) = 2/3 = (1/2)|\phi|(\mathbb{R})$.

8. (25 points). Let P_μ denote the distribution of a $N(\mu, 1)$ random variable X on $\mathbb{R}$: thus P_μ has density with respect to Lebesgue measure λ given by $(dP_\mu/d\lambda)(x) = \phi(x - \mu)$ where $\phi(x) = (2\pi)^{-1/2} \exp(-x^2/2)$.
(a) Show that $P_\mu \ll P_0$.
(b) Find the Radon-Nikodym derivative dP_μ/dP_0.

Solution: (a) Now

$$\begin{aligned} P_\mu(A) &= \int_A \phi(x - \mu) d\lambda(x) = \int_A \frac{\phi(x - \mu)}{\phi(x)} \phi(x) d\lambda(x) \\ &= \int_A \exp(\mu x - \mu^2/2) dP_0(x). \end{aligned}$$

Thus $P_\mu \ll P_0$.
(b) The calculation in (a) shows that $(dP_\mu/dP_0)(x) = \exp(\mu x - \mu^2/2)$.