

Statistics 521, Problem Set 9

Wellner; 11/20/2019

Reading:

- Shorack, PfS, Chapter 7, pages 123 - 129;
Shorack, PfS, Chapter 8, pages 147 - 174.

Due: Wednesday, November 27, 2019

1. PfS, Exercise 7.1.1, page 124:
 - (a) Show that $P(AB) = P(A)P(B)$ if and only if $\{\emptyset, A, A^c, \Omega\}$ and $\{\emptyset, B, B^c, \Omega\}$ are independent σ -fields.
 - (b) Show that A_1, \dots, A_n are independent if and only if (for each $k = 1, \dots, n$,

$$P(A_{i_1} \dots A_{i_k}) = \prod_{j=1}^k P(A_{i_j}) \quad \text{whenever } 1 \leq i_1 < \dots < i_k \leq n.$$

2. Give an example of two collections of sets \mathcal{A}_1 and \mathcal{A}_2 that are independent but the generated σ -fields are not independent.
3. Show that if X_n is any sequence of random variables, there are constants $c_n \rightarrow \infty$ so that $X_n/c_n \rightarrow_{a.s.} 0$.
4. Show that if $P(A_n) \rightarrow 0$ and $\sum_{n=1}^{\infty} P(A_n \cap A_{n+1}^c) < \infty$, then $P(A_n \text{ i.o.}) = 0$.
5. Let X_1, X_2, \dots be independent. Show that $\sup_{n \geq 1} X_n < \infty$ almost surely if and only if $\sum_{n=1}^{\infty} P(X_n > M) < \infty$ for some $M < \infty$.
6. Suppose that X_1, X_2, \dots are independent with $P(X_n > x) = x^{-r}$ for all $x \geq 1$ and $n = 1, 2, \dots$ with $r > 0$. Show that $\limsup_{n \rightarrow \infty} (\log X_n) / \log n = c$ almost surely for some number c , and find c .

7. **Optional bonus problem:** PfS, Exercise 5.3.1, page 94.

Prove that:

- (a) A function $\underline{X} = (X_1, X_2, \dots) : \Omega \rightarrow R^\infty$ is $\mathcal{B}_\infty - \mathcal{A}$ -measurable if and only if each X_n is $\mathcal{B} - \mathcal{A}$ -measurable.
 (b) If $\underline{X} = (X_1, X_2, \dots)$ is $\mathcal{B}_\infty - \mathcal{A}$ -measurable and if (i_1, i_2, \dots) is an arbitrary sequence of integers, then $\underline{Y} \equiv (X_{i_1}, X_{i_2}, \dots)$ is $\mathcal{B}_\infty - \mathcal{A}$ -measurable.

8. **Optional bonus problem:** PfS, Exercise 7.1.3, page 126.

If $\mathcal{C}_1, \dots, \mathcal{C}_n$ are independent classes of events and if each \mathcal{C}_i is a $\bar{\pi}$ -system, then $\sigma[\mathcal{C}_1], \dots, \sigma[\mathcal{C}_n]$ are independent σ -fields.

9. **Optional bonus problem:** Let X_1, X_2, \dots be independent with $P(X_n = 1) = p_n$ and $P(X_n = 0) = 1 - p_n$. Show that: (i) $X_n \rightarrow_p 0$ if and only if $p_n \rightarrow 0$, and $X_n \rightarrow_{a.s.} 0$ if and only if $\sum_n p_n < \infty$.

10. **Optional bonus problem:** (a) Suppose that X_1, X_2, \dots are random variables with mean 0, $EX_j^2 = 1$, and $E(X_i X_j) = 0$ for all $i \neq j$, and let $S_n \equiv X_1 + \dots + X_n$. Show that $S_n/n^\alpha \rightarrow_{a.s.} 0$ for any $\alpha > 1$.
 (b) Suppose that X_1, X_2, \dots are random variables with mean 0, $E(X_i X_j) = 0$ for all $i \neq j$, and $\sup_j EX_j^2 < \infty$. Show that $S_n/n^\alpha \rightarrow_p 0$ for any $\alpha > 1/2$.

11. **Optional bonus problem:** Suppose $U(\omega) = \omega$ for $(\Omega, \mathcal{A}, P) = ((0, 1], \mathcal{B}_{(0,1]}, \lambda)$ where λ is Lebesgue measure (restricted to $(0, 1]$). Thus $U \sim \text{Uniform}(0, 1)$. Define

$$T(\omega) = \begin{cases} 2\omega, & 0 < \omega \leq 1/2, \\ 2\omega - 1, & 1/2 < \omega \leq 1, \end{cases} \quad X_1(\omega) = \begin{cases} 0, & 0 < \omega \leq 1/2, \\ 1, & 1/2 < \omega \leq 1, \end{cases}$$

and, for $i \geq 2$, $X_i(\omega) = X_1(T^{i-1}\omega)$. It follows that

$$\sum_{i=1}^n \frac{X_i(\omega)}{2^i} < \omega \leq \sum_{i=1}^n \frac{X_i(\omega)}{2^i} + \frac{1}{2^n}$$

and the X_i 's give the diadic (non-terminating expansion) representation of U : $U(\omega) = \sum_{i=1}^{\infty} 2^{-i} X_i(\omega)$. Show that X_1, X_2, \dots are independent random variables.

[Hint: see Billingsley, *Probability and Measure*, pages 1-5 and A31, page 572.]