

## Statistics 521, Problem Set 8

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**Reading:** Shorack, PfS, Chapter 5, pages 87 - 97.  
Shorack, PfS, Chapter 6, pages 103 - 113.  
Shorack, PfS, Chapter 7, pages 123 - 129.

**Due:** Wednesday, November 20, 2019

- PfS Exercise 6.4.3, page 114. Prove just the parts of these formulas involving  $F$ , not the parts involving  $F^{-1}$ . You may also use Fubini's theorem directly. That is, show that:
  - If  $X \geq 0$  has d.f.  $F$ , then
$$\int_0^\infty P(X > x)dx = E(X) = \int_0^\infty (1 - F(x))dx.$$
  - If  $E|X| < \infty$  then
$$E(X) = -\int_{-\infty}^0 F(x)dx + \int_0^\infty (1 - F(x))dx.$$
  - Let  $r > 0$ . If  $X \geq 0$ , then
$$\int_0^\infty P(X^r > x)dx = E(X^r) = \int_0^\infty rx^{r-1}(1 - F(x))dx.$$
- Prove the two formulas in (17), PfS page 113: if  $X \geq 0$  is integer valued, then  $E(X) = \sum_{k=1}^\infty P(X \geq k)$  and  $E(X^2) = \sum_{k=1}^\infty (2k - 1)P(X \geq k)$ .
- PfS Exercise 4.9, page 114: For any distribution function  $F$  on  $\mathbb{R}$  we have  $\int \{F(x + \theta) - F(x)\}dx = \theta$  for each  $\theta \geq 0$ .
- PfS, Exercise 5.1.4, page 91.  
Let  $\mathcal{X} = [0, 1]$ ,  $\mathcal{Y} = (1, \infty)$  both equipped with the Borel sets and Lebesgue measure. Let  $f(x, y) = e^{-xy} - 2e^{-2xy}$ . Show that
  - $\int_0^1 (\int_1^\infty f(x, y)dy)dx = \int_0^1 x^{-1}(e^{-x} - e^{-2x})dx$  exists and is  $> 0$ .
  - $\int_1^\infty (\int_0^1 f(x, y)dx)dy = \int_1^\infty y^{-1}(e^{-2y} - e^{-y})dy$  exists and is  $< 0$ .
- Optional bonus problem:** PfS Exercise 4.11, page 114:
  - Show that  $\int_0^\infty \{P(|X| > x)\}^{1/2}dx < \infty$  implies  $E(X^2) < \infty$ .
  - Show that  $\int_0^\infty \{P(|X| > x)\}^{1/2}dx \leq \frac{r}{r-2}\|X\|_r$  for  $r > 2$  so that the integral on the left is finite whenever  $X \in \mathcal{L}_r$  for any  $r > 2$ . If  $\int_0^\infty \{P(|X| > x)\}^{1/2}dx < \infty$  then we say that  $X \in \mathcal{L}_{2,1}$ ; this condition arises in connection with optimal transportation inequalities for empirical processes and in multiplier and bootstrap CLTs.

6. **Optional bonus problem:** Show that  $f(x, y) = e^{-xy} \sin(x)$  is integrable with respect on Lebesgue measure on  $\mathbb{R}^2$  in the strip  $0 < x < a$ ,  $0 < y$ . Perform the double integral in the two different orders to find that

$$\int_0^a \frac{\sin(x)}{x} dx = \frac{\pi}{2} - \cos(a) \int_0^\infty \frac{e^{-ay}}{1+y^2} dy - \sin(a) \int_0^\infty \frac{ye^{-ay}}{1+y^2} dy.$$

Use the inequality  $1 + y^2 \geq 1$  to obtain the bound

$$\left| \int_0^a \frac{\sin(x)}{x} dx - \frac{\pi}{2} \right| \leq \frac{2}{a}.$$

Letting  $a \rightarrow \infty$  yields  $\int_0^\infty x^{-1} \sin(x) dx = \pi/2$ .

7. **Optional bonus problem:** (A bivariate exponential distribution). Let  $Y_1 \sim \text{Exponential}(\mu_1)$ ,  $Y_2 \sim \text{Exponential}(\mu_2)$ , and  $W \sim \text{Exponential}(\tau)$  be independent.
- Let  $X_1 \equiv \min\{Y_1, W\}$ ,  $X_2 \equiv \min\{Y_2, W\}$ . Find the joint survival function  $\bar{F}(x_1, x_2) \equiv P(X_1 > x_1, X_2 > x_2)$  of  $(X_1, X_2)$ .
  - Find  $P(X_1 = X_2)$ .
  - Use (b) to show that the joint probability distribution  $Q \equiv P_{X_1, X_2}$  of  $(X_1, X_2)$  is not absolutely continuous with respect to Lebesgue measure  $\lambda_2$  on  $\mathbb{R}^2$ . Identify the singular component of the joint distribution  $Q$  by computing  $P(X_1 = X_2 > z)$  for  $z \geq 0$ .
  - What is the density of  $Q_{ac}$ , the absolutely continuous component of  $Q$  with respect to  $\lambda_2$ ?