**Statistics 521, Problem Set 7**

Wellner; 11/6/2019

**Reading:**

- Shorack, PfS, Chapter 4, sections 4.1 - 4.4, pages 65 - 85;
- Durrett, *Probability*, Appendix pages 403 - 407;
- Shorack, PfS, Chapter 5, sections 5.1 - 5.3, pages 87 - 101.

**Due:** Wednesday, November 13, 2019.

1. PfS, Exercise 4.1.2, page 67: Identify $\phi^+$, $\phi^-$, $|\phi|$ and $|\phi|(\Omega)$ in the context of the prototypical situation of example 4.1.1, page 66: in particular $\phi(A) = \int_A X d\mu$ where $X$ is measurable. Be sure to specify $\Omega^+$ and $\Omega^-$ as in the Jordan - Hahn decomposition.

2. Let $\mu$ be a sigma-finite measure and let $\nu$ be a finite measure on $(\Omega, \mathcal{A})$. Set $\phi = \mu - \nu$; i.e. define $\phi : \mathcal{A} \to (-\infty, \infty]$ by $\phi(A) = \mu(A) - \nu(A)$.
(a) Show that $\phi$ is a signed measure.
(b) Show that
$$\phi(A) = \int_A (f - g) d(\mu + \nu)$$
for some measurable functions $f$ and $g$, $g \in \mathcal{L}_1(\mu + \nu)$. Thus $\phi$ can be written in the canonical form of the signed measure discussed in example 4.1.1 and 1 above.
(c) Apply the results of PfS Exercise 4.1.2, page 67, to $\phi$: compute $\phi^+$, $\phi^-$, $|\phi|$, and $|\phi|(\Omega)$, assuming for the latter that $\mu$ is also a finite measure.

3. For probability measures $P$ and $Q$ on $(\Omega, \mathcal{A})$, define
$$d_{TV}(P, Q) = \sup_{A \in \mathcal{A}} |P(A) - Q(A)| .$$
We showed in problem 3, problem set #6 that $d_{TV}(P, Q) = (1/2) \int |p - q| d\mu$ for any measure $\mu$ dominating both $P$ and $Q$; i.e. $P << \mu$, $Q << \mu$.
(a) Show that $d_{TV}(P, Q)$ does not depend on the choice of $\mu$.
(b) Use the results of problem 1, part (c) to show that $d_{TV}(P, Q) = (1/2)|P - Q|(\Omega)$.

4. Suppose that $\phi$ is a sigma-finite signed measure and $X \in L_1(|\phi|)$. The integral $\int X d\phi$ is defined by

$$\int X d\phi = \int X d\phi^+ - \int X d\phi^- .$$

Show that $|\int X d\phi| \le \int |X| d|\phi|$.

5. PfS, Exercise 4.2.3, page 73: Flip a coin. If heads results, let $X$ be a Uniform$(0,1)$ random variable; if tails results, let $X$ be a Poisson$(\lambda)$ random variable. The resulting distribution of $X$ on $\mathbb{R}$ is labeled $\phi$.
(a) Let $\mu$ denote Lebesgue measure on $\mathbb{R}$. Find the Lebesgue decomposition of $\phi$ with respect to $\mu$; that is, write $\phi = \phi_{ac} + \phi_s$.
(b) Let $\nu$ be counting measure on $\{0, 1, 2, \ldots\}$. Find the Lebesgue decomposition of $\phi$ with respect to $\nu$.

6. PfS, Exercise 4.4.3, page 84: Let $F$ be $\nearrow$, right-continuous and bounded on $\mathbb{R}$ with $F(-\infty) = 0$. Define $\mu_F$ via $\mu_F((a, b]) = F(b) - F(a)$ for all $a < b$. Show that $\mu_F << \lambda$ if and only if $F$ is an absolutely continuous function on $\mathbb{R}$ (in the sense of PfS, Definition 4.4.2, page 80).

7. **Bonus problem:**
(a) Let the Hellinger distance $H(P, Q)$ be defined for probability measures $P, Q$ on a measurable space $(\Omega, \mathcal{A})$ by

$$H^2(P, Q) \equiv \frac{1}{2} \int \{\sqrt{p} - \sqrt{q}\}^2 d\mu$$

where $p = dP/d\mu$ and $q = dQ/d\mu$ and $\mu$ is any measure dominating both $P$ and $Q$: $P << \mu$ and $Q << \mu$. Show that $H(P, Q)$ does not depend on the choice of $\mu$.
(b) Show that

$$H^2(P, Q) \le d_{TV}(P, Q) \le cH(P, Q)$$

for some absolute constant $c$. Find $c$ explicitly.