Statistics 521, Problem Set 6

Wellner; 10/30/2019

Reminders:

- Midterm exam: Friday, November 1.

Reading:

- Shorack, PfS, Chapter 4, sections 4.1 - 4.4, pages 65 - 85;
- Durrett, *Probability*, Appendix, pages 403 - 407.

Due: Wednesday, November 6, 2019.

1. (a) Give an example of a sequence of random variables X_n, X (all defined on a common probability space $(\Omega, \mathcal{A}, P)$) satisfying $X_n \rightarrow_{a.s.} X$, but $E(X_n) \nrightarrow E(X)$.
   (b) Give an example of a sequence of non-negative random variables X_n, X on a common probability space satisfying $E(X_n) \rightarrow E(X)$ but $X_n \nrightarrow_{a.s.} X$.
   (c) Give an example of a sequence of random variables X_n, X satisfying $X_n \rightarrow_d X$, but $X_n \nrightarrow_{p,a.s.,1} X$.

2. PfS, Exercise 3.5.7, page 61, modified as follows: Suppose that $f_0, f_1, \ldots$ are ≥ 0, defined on a sigma-finite measure space $(\Omega, \mathcal{A}, \mu)$. (a) Suppose that $\int_\Omega f_n d\mu = 1$ for $n = 0, 1, \ldots$, and $f_n \rightarrow_{a.e.} f_0$ with respect to μ. Show that

$$\sup_{A \in \mathcal{A}} \left| \int_A f_n d\mu - \int_A f_0 d\mu \right| \rightarrow 0 \qquad \text{as} \ \ n \rightarrow \infty .$$

   (b) Show that the conclusion of (a) holds if just $f_n \rightarrow_\mu f_0$ and $\int_\Omega f_n d\mu \rightarrow \int_\Omega f_0 d\mu$.

3. Let $X_{n1}, \ldots, X_{nn}$ be independent, $X_{nk} \sim$ Bernoulli(p_{nk}), and let $Y_n \sim$ Poisson$(\sum_{k=1}^n p_{nk})$. Let P_n be the distribution of $\sum_{k=1}^n X_{nk}$ and let Q_n be the distribution of Y_n. Show that

$$d_{TV}(P_n, Q_n) \equiv \sup_{A \in \mathcal{B}} |P(S_n \in A) - P(Y_n \in A)| \leq \sum_{k=1}^n p_{nk}^2 .$$

Note that when $p_{nk} = p_n \to 0$ for all k and $np_n \to \lambda$, then $\sum_{k=1}^n p_{nk}^2 = np_n^2 = (np_n)^2/n = O(n^{-1})$.

Hint: Construct S_n and Y_n on a common probability space as follows: let $T_{nk} \sim$ Poisson(p_{nk}), $k = 1, \ldots, n$ be independent, and let $Z_{nk} \sim$ Bernoulli$(1-(1-p_{nk})e^{p_{nk}})$, $k = 1, \ldots, n$ be independent and independent of the T_{nk}'s. Define $X_{nk} = 1_{[T_{nk} \geq 1]} + 1_{[T_{nk}=0]}1_{[Z_{nk}=1]}$. Set $S_n = \sum_{k=1}^n X_{nk}$, $Y_n = \sum_{k=1}^n T_{nk}$. Check that $X_{nk} \sim$ Bernoulli(p_{nk}), $Y_n \sim$ Poisson$(\sum_1^n p_{nk})$, and

$$\begin{aligned} P(T_{nk} = 0, X_{nk} = 1) &= e^{-p_{nk}} - (1 - p_{nk}) \\ P(T_{nk} \geq 1, X_{nk} = 0) &= 0, \quad P(T_{nk} \geq 2) = 1 - e^{-p_{nk}} - p_{nk}e^{-p_{nk}}. \end{aligned}$$

Show that

$$d_{TV}(P_n, Q_n) \leq P(S_n \neq Y_n) \leq \sum_{k=1}^n P(X_{nk} \neq T_{nk}) \leq \sum_{k=1}^n p_{nk}^2.$$

4. Let $Z \sim N(0,1)$ and let $Y_r \sim \chi_r^2$ for $r > 0$. Thus $X_r \equiv Z/\sqrt{Y_r/r} \sim t_r$, the Student $t-$ distribution with r "degrees of freedom" with density

$$f_r(t) \equiv C_r \left(1 + \frac{t^2}{r}\right)^{-(r+1)/2}$$

where $C_r \equiv \Gamma((r+1)/2)/(\sqrt{\pi r}\Gamma(r/2)$. Let P_r denote the corresponding probability measure on $\mathbb{R}$.
(a) Show that $Y_r/r \to_p 1$ as $r \to \infty$.
(b) Show that $X_r \to_d Z \sim N(0,1)$ as $r \to \infty$.
(c) Show that $f_r(t) \to \phi(t) \equiv (2\pi)^{-1/2}e^{-t^2/2}$.
(d) Use the result of (c) to show that $d_{TV}(P_r, P_Z) \to 0$ as $r \to \infty$.

5. **Optional bonus problem:** Let X and Y be non-negative random variables.
(a) Show that Hölder's inequality can be rewritten as

$$E(X^{1/r}Y^{1/s}) \leq (EX)^{1/r} \cdot (EY)^{1/s} \quad \text{where} \quad r^{-1} + s^{-1} = 1.$$

(b) Show that the function $g(x,y) = x^{1/r}y^{1/s}$ is a concave function of (x,y); i.e. show that $(\partial^2/\partial x^2)f \leq 0$, $(\partial^2/\partial y^2)f \leq 0$, and

$$\left\{\frac{\partial^2}{\partial x \partial y}f(x,y)\right\}^2 - \frac{\partial^2}{\partial x^2}f(x,y)\frac{\partial^2}{\partial y^2}f(x,y) \leq 0.$$

(c) Use the bivariate version of Jensen's inequality to prove the form of Hölder's inequality given in (a).

6. **Optional bonus problem:** Let $X_1, X_2, \ldots$ be i.i.d. Exponential(λ) random variables (with distribution function $F(x) = 1 - \exp(-\lambda x)$, $x \ge 0$), and let $Y_n \equiv X_{n:n} \equiv \max_{1 \le i \le n} X_i$.
   (a) Find the distribution function F_{Y_n} of Y_n and find a sequence b_n so that $Z_n \equiv Y_n - b_n \to_d Z$; identify the distribution function F_Z of Z.
   (b) Find the density function f_{Z_n} of Z_n and show that $f_{Z_n}(z) \to f_Z(z)$ for all $z \in \mathbb{R}$.
   (c) Use (b) and Scheffé's theorem to show that

$$d_{TV}(P_{Z_n}, P_Z) = \frac{1}{2} \int_0^\infty |f_{Z_n}(z) - f_Z(z)| dz \to 0.$$

7. **Optional bonus problem:** Suppose that $g : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ is convex. Show that $h(x,t) \equiv t g(x/t)$ is a convex function on $\mathbb{R}^d \times (0, \infty)$.
   Hint: First show that $h(cx, ct) = c h(x,t)$ for any $c > 0$, $(x,t) \in \mathbb{R}^d \times (0,\infty)$, and hence $t^{-1} h(x,t) = h(x/t, 1)$.