Statistics 521, Problem Set 5
Wellner; 10/23/2019

Reading: Shorack, PfS, Chapter 3, section 3.5, pages 52-63;
Shorack, PfS, Chapter 4, sections 4.1 - 4.4, pages 65-79.
Durrett, *Probability*, pages 404 - 407.

Reminder, Make-up Lecture: Friday, 25 October, 8:30-9:20, Low 101
Reminder, Mid-term exam: Friday, 1 November, 11:30-12:20, Low 106.

Due: Wednesday, October 30, 2019.

1. PfS, Exercise 3.4.2, page 48: Show that $\rho = 1$ if and only if $X - \mu_X = a(Y - \mu_Y)$ for some $a > 0$; and $\rho = -1$ if and only if $X - \mu_X = a(Y - \mu_Y)$ for some $a < 0$. Thus ρ measures linear dependence, not dependence.

2. PfS, Exercise 3.4.3, page 48: (Littlewood's inequalities) Let $\mu_r \equiv E|X|^r$. For $r \ge s \ge t \ge 0$ we have $\mu_r^{s-t}\mu_t^{r-s} \ge \mu_s^{r-t}$. In particular, $\mu_2^3 \le \mu_1^2 \mu_4$. Hint: write $E|X|^s = E|X|^{\lambda s} \cdot |X|^{(1-\lambda)s}$ and apply Hölder's inequality.

3. Suppose that $\epsilon_1, \ldots, \epsilon_n$ are i.i.d. random variables with $P(\epsilon_i = \pm 1) = 1/2$, and let $a_i \in \mathbb{R}$, $i = 1, \ldots, n$. Khintchine's inequalities say that for each $p > 0$

$$A_p \left(\sum_{i=1}^n a_i^2 \right)^{1/2} \le \left(E|\sum_{i=1}^n a_i \epsilon_i|^p \right)^{1/p} \le B_p \left(\sum_{i=1}^n a_i^2 \right)^{1/2} .$$

for some constants A_p and B_p. Prove the above inequalities when $p = 1$. **Hint:** The inequality on the right side is easy. Use the previous exercise to prove the inequality on the left side by showing that for $Z \equiv \sum_{i=1}^n a_i \epsilon_i$, we have $E|Z|^4 \le 3(E(Z^2))^2$.

4. PfS, Exercise 3.5.3, page 55: Consider a probability measure P. (a) Let $Y \ge 0$ have df F. Show that $EY = \int_0^\infty P(Y \ge y) dy = \int_0^\infty [1 - F(y)] dy$. [Hint: prove the claimed formula for simple functions by summing by

parts; and then the full claim follows from the MCT. A different proof to come later will use Fubini's theorem.]
(b) use the result of (a) to show that for $Y \geq 0$ and $\lambda \geq 0$ we have

$$\int_{[Y \geq \lambda]} Y\, dP = \lambda P(Y \geq \lambda) + \int_\lambda^\infty P(Y \geq y) dy.$$

Draw a picture to illustrate this.
(c) Suppose there is a $Y \in \mathcal{L}_1$ such that $P(|X_n| \geq y) \leq P(Y \geq y)$ for all $y > 0$ and all $n \geq 1$. Then use (b) to show that $\{X_n : \ n \geq 1\}$ is uniformly integrable.

5. (a) Show that if $|X_n| \leq Y$ and Y is integrable, then $\{X_n\}$ is uniformly integrable.
(b) Let $U \sim$Uniform$(0,1)$, and let $X_n \equiv (n/\log n) 1_{[0,1/n]}(U)$ for $n \geq 3$. Show that $\{X_n\}$ is uniformly integrable and $\int X_n dP \to 0$ even though they are not dominated by any integrable rv Y.
(c) Let $Z_n = n1_{[0,1/n]}(U) - n1_{[1/n,2/n]}(U)$. Show that $\{Z_n\}$ is not uniformly integrable, but that $\int Z_n dP \to 0$.

6. **Optional bonus problem:** PfS, Exercise 3.4.6, page 50 (qualified by "for all $\epsilon \geq 1$"): (a) Let $T \sim$ Binomial(n,p), so $P(T = k) = \binom{n}{k} p^k (1-p)^{n-k}$ for $0 \leq k \leq n$. The measure associated with T has mean np and variance $np(1-p)$. Then use inequality 4.6 with $g(x) = \exp(rx)$ and $r > 0$ to show that

$$P(T/n \geq px) \leq \exp(-nph(x)), \qquad \text{where } h(x) \equiv x(\log(x) - 1) + 1$$

for each $x \geq 1$. [Hint: It helps to use $T \stackrel{d}{=} \sum_1^n X_i$ where $X_i \sim$ Bernoulli(p) are independent, and then apply Theorem 7.1.1 (page 124).]
(b) Show that the inequality in (a) implies that

$$\begin{aligned} P(\sqrt{n}(T/n - p) > \lambda) \ &\leq \ \exp\left(-nph(1 + \frac{\lambda}{\sqrt{np}})\right) \\ &= \ \exp\left(-\frac{\lambda^2}{2p}\psi(\frac{\lambda}{\sqrt{np}})\right). \end{aligned}$$

where $\psi(y) \equiv 2y^{-2}h(1 + y)$.
(c) Plot the functions $y \mapsto h(1 + y)$ and $\psi(y)$.