Statistics 521, Problem Set 4

Wellner; 10/16/2019

Reading:

Shorack, PfS, Chapter 3, sections 3.1-3.4, pages 37-61;
Durrett, *Probability*, sections 1.4-1.6, pages 15-36;
sections 3.1-3.2, pages 98-106.

Reminder: Make-up lecture: Friday, 25 October, 8:30 - 9:20, Low 106.
Reminder: Midterm exam: Friday, November 1.

Due: Wednesday, October 23, 2019.

1. PfS, Exercise 2.3.4, page 32: (a) Suppose that $\mu(\Omega) < \infty$ and g is continuous a.e. μ_X (that is, g is continuous except perhaps on a set of μ_X measure 0). Then $X_n \rightarrow_\mu X$ implies that $g(X_n) \rightarrow_\mu g(X)$.
(b) Let g be uniformly continuous on the real line. Then $X_n \rightarrow_\mu X$ implies that $g(X_n) \rightarrow_\mu g(X)$. (Here $\mu(\Omega) = \infty$ is allowed.)

2. PfS, Exercise 3.2.1, page 42: Show that $X \geq 0$ and $\int X d\mu = 0$ implies $\mu([X > 0]) = 0$.

3. PfS, Exercise 3.2.2, page 42: Show that

$$\int_A X d\mu = \begin{cases} = 0, \\ \geq 0, \end{cases} \qquad \text{for all } A \in \mathcal{A} \ \text{ implies } \ X = \begin{cases} = 0 & \text{a.e.,} \\ \geq 0 & \text{a.e.} \end{cases}$$

4. PfS, Exercise 3.2.4, page 43. Let $Y \equiv g(X)$ in the context of Theorem 3.2.6 (the "Theorem of the unconscious statistician"). Show that the second equality holds in:

$$\int_{X^{-1}(g^{-1}(B))} g(X(\omega)) d\mu(\omega) = \int_{g^{-1}(B)} g(x) d\mu_X(x) = \int_B y d\mu_Y(y) \quad \text{for } B \in \overline{\mathcal{B}}$$

where μ_Y is the induced measure of Y on $(\overline{R}, \overline{\mathcal{B}})$.

5. (i) PfS, Exercise 3.3, page 45, part (a).
(ii) Suppose that μ is Lebesgue measure on the unit interval $[0,1]$ and that $(a,b) = (0,1)$ in Exercise 3.3. If $X(t,\omega) = 1_{[\omega \le t]}$, then for each t, $(\partial/\partial t)X(t,\omega) = 0$ almost everywhere. But $\int X(t,\omega)d\mu(\omega)$ does not differentiate to 0. Why is this not a contradiction?

6. **Bonus problem:** (See PfS Example 1.1, page 105; Durrett Example 1.2.4.) The Cantor singular distribution function F is the function $F : [0,1] \to [0,1]$ defined as follows: $F(x) = 1/2$ for $x \in (1/3, 2/3)$; $F(x) = 1/4$ for $x \in (1/9, 2/9)$ and $F(x) = 3/4$ for $x \in (7/9, 8/9)$; $\ldots$; $F(x) = 1/2^n, 3/2^n, 5/2^n, \ldots$ on the successive intervals removed from C_{n-1} in the construction of C_n. Thus F is defined on the open set $[0,1] \setminus C$, is nondecreasing, and has values in $[0,1]$. Extend it to all of $[0,1]$ by letting $F(0) = 0$, and setting

$$F(x) \equiv \sup\{F(y) : y \in [0,1] \setminus C \ \text{ and } \ y < x\}$$

for $x \in C$ and $x \neq 0$.
(i) Show that F is non-decreasing and continuous with $F(0) = 0$ and $F(1) = 1$. Because F is continuous, its range is all of $[0,1]$.
(ii) Now the inverse (or quantile) function F^{-1} of F defined by

$$F^{-1}(y) \equiv \inf\{x \in [0,1] : \ F(x) \ge y\}$$

is one-to-one (injective) and $F^{-1}([0,1]) \subset C$. Show that F^{-1} is Borel-measurable.
(iii) Show that the lengths of the "flat spots" in F sum to 1.