

## Statistics 521, Problem Set 3

Wellner; 10/09/19

### Reading:

Shorack, PfS, Chapter 2, pages 21-36;  
Durrett, *Probability*, sections 1.2-1.6, pages 9-36.

**Due:** Wednesday, October 16, 2019.

1. PfS, Exercise 2.2.1, page 28:  
Suppose that  $(\Omega, \mathcal{A}) = (R_2, \mathcal{B}_2)$  where  $\mathcal{B}_2$  denotes the  $\sigma$ -field generated by all open subsets of the plane. Recall that this  $\sigma$ -field contains all sets of the form  $B \times R$  and  $R \times B$  for all  $B \in \mathcal{B}$  where  $B_1 \times B_2 \equiv \{(r_1, r_2) : r_1 \in B_1, r_2 \in B_2\}$ . Now define measurable transformations  $X_1(r_1, r_2) = r_1$  and  $X_2(r_1, r_2) = r_2$ . Then define  $Z_1 \equiv \sqrt{X_1^2 + X_2^2}$  and  $Z_2 \equiv \text{sign}(X_1 - X_2)$  where  $\text{sign}(r) = 1, 0, -1$  according as  $r$  is  $> 0, = 0, < 0$ . Give geometric descriptions of the  $\sigma$ -fields  $\mathcal{F}(Z_1)$ ,  $\mathcal{F}(Z_2)$ , and  $\mathcal{F}(Z_1, Z_2)$ .
2. PfS, Exercise 2.2.2, page 28:  
Suppose that  $\mathcal{C}$  is a  $\bar{\pi}$ -system. Suppose that  $\mathcal{V}$  is a vector space of functions with:
  - (i)  $1_C \in \mathcal{V}$  for all  $C \in \mathcal{C}$ .
  - (ii) If  $A_n \in \mathcal{V}$  satisfy  $A_n \nearrow A$ , then  $A \in \mathcal{V}$ .
  - (a) Show that  $1_A \in \mathcal{V}$  for every  $A \in \sigma[\mathcal{C}]$ .
  - (b) Show that every simple function

$$\sum_1^m x_i 1_{A_i} \text{ is in } \mathcal{V}$$

whenever  $m \geq 1$ ,  $x_i \in R$ , and  $\sum_1^m A_i = \Omega$  with  $A_i \in \sigma[\mathcal{C}]$ .

(c) Suppose further that  $X_n \nearrow X$  for  $X_n$ 's as in (b) implies that  $X \in \mathcal{V}$ . Show that  $\mathcal{V}$  contains all  $\sigma[\mathcal{C}]$ -measurable functions.

3. PfS, Exercise 2.3.1, page 29:  
Let  $X_1, X_2, \dots$  denote measurable functions from  $(\Omega, \mathcal{A}, \mu)$  to  $(\bar{R}, \bar{\mathcal{B}})$ .
  - (a) If  $X_n \rightarrow_{a.e.} X$ , then  $X = \tilde{X}$  a.e. for some measurable  $\tilde{X}$ .
  - (b) If  $X_n \rightarrow_{a.e.} X$  and  $\mu$  is complete, then  $X$  itself is measurable.

4. PfS, Exercise 2.3.2, page 31:
  - (a) Show that in general  $\rightarrow_{\mu}$  does not imply  $\rightarrow_{a.e.}$ .
  - (b) Give an example with  $\mu(\Omega) = \infty$  where  $\rightarrow_{a.e.}$  does not imply  $\rightarrow_{\mu}$ .
  
5. PfS, Exercise 2.3.3, page 32.  
 show that  $X_n \rightarrow_{\mu} X$  if and only if  $X_n - X_m \rightarrow_{\mu} 0$ .
  
6. **Bonus problem 1:** Prove Slutsky's theorem (Theorem 4.1, PfS page 34): If  $X_n \rightarrow_d X$  and random variables  $Y_n$  and  $Z_n$  satisfy  $Y_n \rightarrow_p Y$  and  $Z_n \rightarrow_p b$ , then  $Y_n X_n + Z_n \rightarrow_d aX + b$ .
  
7. **Bonus problem 2:** Suppose that  $(\Omega, \mathcal{A}, P) = ([0, 1], \mathcal{B}^{\infty}, P)$  where  $P \equiv \mu$  is Lebesgue measure on the Borel subsets of  $[0, 1]$ .
  - (a) What is the distribution function  $F$  corresponding to  $P = \mu$ ?
  - (b) If  $U(\omega) = \omega$  for  $\omega \in \Omega = [0, 1]$ , compute  $P(U \leq u) = P(\{\omega : U(\omega) \leq u\})$  for  $u \in [0, 1]$ .
  - (c) If  $g(u) \equiv u^2$  for  $u \in [0, 1]$ , compute  $P(g(U) \leq v)$  for  $v \in [0, 1]$ .