Statistics 521, Problem Set 3
Wellner; 10/09/19

Reading:

Shorack, PfS, Chapter 2, pages 21-36;
Durrett, *Probability*, sections 1.2-1.6, pages 9-36.

Due: Wednesday, October 16, 2019.

1. PfS, Exercise 2.2.1, page 28:
Suppose that $(\Omega, \mathcal{A}) = (R_2, \mathcal{B}_2)$ where $\mathcal{B}_2$ denotes the $\sigma-$field generated by all open subsets of the plane. Recall that this $\sigma-$field contains all sets of the form $B \times R$ and $R \times B$ for all $B \in \mathcal{B}$ where $B_1 \times B_2 \equiv \{(r_1, r_2) : r_1 \in B_1, r_2 \in B_2\}$. now define measurable transformations $X_1(r_1, r_2) = r_1$ and $X_2(r_1, r_2) = r_2$. Then define $Z_1 \equiv \sqrt{X_1^2 + X_2^2}$ and $Z_2 \equiv \text{sign}(X_1 - X_2)$ where $\text{sign}(r) = 1, 0, -1$ according as r is $> 0, = 0, < 0$. Give geometric descriptions of the $\sigma-$fields $\mathcal{F}(Z_1)$, $\mathcal{F}(Z_2)$, and $\mathcal{F}(Z_1, Z_2) \equiv \sigma[\mathcal{F}(Z_1), \mathcal{F}(Z_2)]$.

2. PfS, Exercise 2.2.2, page 28:
Suppose that $\mathcal{C}$ is a $\overline{\pi}-$system. Suppose that $\mathcal{V}$ is a vector space of functions with:
(i) $1_C \in \mathcal{V}$ for all $C \in \mathcal{C}$.
(ii) If $A_n \in \mathcal{V}$ satisfy $A_n \nearrow A$, then $A \in \mathcal{V}$.
(a) Show that $1_A \in \mathcal{V}$ for every $A \in \sigma[\mathcal{C}]$.
(b) Show that every simple function
$$\sum_1^m x_i 1_{A_i} \quad \text{is in} \quad \mathcal{V}$$
whenever $m \geq 1$, $x_i \in R$, and $\sum_1^m A_i = \Omega$ with $A_i \in \sigma[\mathcal{C}]$.
(c) Suppose further that $X_n \nearrow X$ for X_n's as in (b) implies that $X \in \mathcal{V}$. Show that $\mathcal{V}$ contains all $\sigma[\mathcal{C}]-$measurable functions.

3. PfS, Exercise 2.3.1, page 29:
Let $X_1, X_2, \ldots$ denote measurable functions from $(\Omega, \mathcal{A}, \mu)$ to $(\overline{R}, \overline{\mathcal{B}})$.
(a) If $X_n \to_{a.e.} X$, then $X = \widetilde{X}$ a.e. for some measurable $\widetilde{X}$.
(b) If $X_n \to_{a.e.} X$ and μ is complete, then X itself is measurable.

4. PfS, Exercise 2.3.2, page 31:
   (a) Show that in general $\to_\mu$ does not imply $\to_{a.e.}$.
   (b) Give an example with $\mu(\Omega) = \infty$ where $\to_{a.e.}$ does not imply $\to_\mu$.

5. PfS, Exercise 2.3.3, page 32.
   show that $X_n \to_\mu X$ if and only if $X_n - X_m \to_\mu 0$.

6. **Bonus problem 1:** Prove Slutsky's theorem (Theorem 4.1, PfS page 34): If $X_n \to_d X$ and random variables Y_n and Z_n satisfy $Y_n \to_p a$ and $Z_n \to_p b$, then $Y_n X_n + Z_n \to_d aX + b$.

7. **Bonus problem 2:** Suppose that $(\Omega, \mathcal{A}, P) = ([0,1], \mathcal{B}^1, P)$ where $P \equiv \mu$ is Lebesgue measure on the Borel subsets $\mathcal{B}^1$ of $[0,1]$.
   (a) What is the distribution function F corresponding to $P = \mu$?
   (b) If $U(\omega) = \omega$ for $\omega \in \Omega = [0,1]$, compute
   $P(U \le u) = P(\{\omega : \ U(\omega) \le u\})$ for $u \in [0,1]$.
   (c) If $g(u) \equiv u^2$ for $u \in [0,1]$, compute $P(g(U) \le v)$ for $v \in [0,1]$.