Statistics 521, Problem Set 2

Wellner; 10/2/2019

Reading: Shorack, PfS, Chapters 1-2, pages 15-20 and 21 - 28.
Durrett, Probability, pages 14-23 and pages 399 - 401.

Due: *Wednesday, October 9*, 2019.

1. (Carried over from problem set # 1). PfS, Exercise 1.1.2, PfS, page 8. We always have $\mu(\liminf A_n) \le \liminf \mu(A_n)$, while $\limsup \mu(A_n) \le \mu(\limsup A_n)$ if $\mu(\Omega) < \infty$.

2. PfS, Exercise 1.1.3, page 9 (and read the proof of the $\pi - \lambda$ theorem, Proposition 1.1.5, pages 9-10).
(a) The minimal $\lambda-$system generated by the class $\mathcal{D}$ is denoted by $\lambda[\mathcal{D}]$. Show that $\lambda[\mathcal{D}]$ is equal to the intersection of all $\lambda-$systems containing $\mathcal{D}$.
(b) A collection $\mathcal{A}$ of subsets of Ω is a $\sigma-$field if and only if it is both a $\pi-$system and a $\lambda-$system.
(c) Let $\mathcal{C}$ be a $\pi-$system and let $\mathcal{D}$ be a $\lambda-$system. Then $\mathcal{C} \subset \mathcal{D}$ implies that $\sigma[\mathcal{C}] \subset \mathcal{D}$.

3. PfS, Exercise 1.2.1, page 15. Let $(\Omega, \mathcal{A}, \mu)$ denote a measure space. Show that

$$\begin{aligned} \widehat{\mathcal{A}}_\mu &\equiv \{A : A_1 \subset A \subset A_2,\ A_1, A_2 \in \mathcal{A}, \mu(A_2 \setminus A_1) = 0\} \\ &= \{A \cup N : A \in \mathcal{A} \text{ and } N \subset (\text{some } B) \in \mathcal{A} \text{ with } \mu(B) = 0\} \\ &= \{A \triangle N : \ A \in \mathcal{A},\ N \subset (\text{some } B) \in \mathcal{A} \text{ with } \mu(B) = 0\} \end{aligned}$$

and is a $\sigma-$field. Show that $(\Omega, \widehat{\mathcal{A}}_\mu, \widehat{\mu})$ is complete.

4. PfS, Exercise 1.2.3, page 16.
Suppose that μ on a field $\mathcal{C}$ is $\sigma-$finite on $\mathcal{C}$ and is extended to $\mathcal{A} = \sigma[\mathcal{C}]$; call the extension μ.
(a) For each $A \in \mathcal{A}$ with $\mu(A) < \infty$ and each $\epsilon > 0$ there exists a set $C = C_\epsilon \in \mathcal{C}$ such that $\mu(A \triangle C) < \epsilon$.
(b) Let μ denote counting measure on the integers. Then $\mathcal{C} = \{C :$

C or C^c is finite} is a field. Determine $\sigma[\mathcal{C}]$. Show that the conclusion of part (a) fails for the set of even integers.

5. PfS, Exercise 1.2.4, page 16. (Nonmeasurable sets). Let Ω consist of the 16 values $1, \ldots, 16$. (Think of them arranged in four rows of four values.) Let

$$\begin{array}{ll} C_1 = \{1,2,3,4,5,6,7,8\}, & C_2 = \{9,10,11,12,13,14,15,16\}, \\ C_3 = \{1,2,5,6,9,10,13,14\}, & C_4 = \{3,4,7,8,11,12,15,16\}. \end{array}$$

Let $\mathcal{C}$ denote the field generated by $\{C_1, C_2, C_3, C_4\}$, and let $\mathcal{A} = \sigma[\mathcal{C}]$.
(a) Show that $\mathcal{A} \equiv \sigma[\mathcal{C}] \neq 2^\Omega$. (Note that 2^Ω contains $2^{16} = 65,536$ sets.)
(b) Let $\mu(C_i) = 1/2$, $1 \leq i \leq 4$, with $\mu(C_1 C_3) = 1/4$. Show $\hat{\mathcal{A}}_\mu = \mathcal{A}$ with $2^4 = 16$ sets.
(c) Let $\mu(C_i) = 1/2$ for $i = 2,3,4$, with $\mu(C_2 C_4) = 0$. Show that $\hat{\mathcal{A}}_\mu$ has $2^{10} = 1024$ sets.
(d) Illustrate Proposition 2.1, PfS page 16 in the context of this exercise.

6. **Optional bonus problem 1:** PfS, Exercise A.1.5, page 428 (or PfS(2000), 9.1.5, page 182): It is shown on page 428 that for $W_m \equiv \inf\{t > 0 : \mathbb{N}(t) = m\}$ where $\mathbb{N}$ is a standard Poisson process on $\mathbb{R}^+$ with intensity ν, we have

$$1 - F_{W_m}(t) = P(W_m > t) = \sum_{k=0}^{m-1} (\nu t)^k e^{-\nu t}/k!.$$

By differentiating both sides of this identity show that W_m has density

$$f_{W_m}(t) = \nu(\nu t)^{m-1} e^{-\nu t}/\Gamma(m) \quad \text{for} \quad t \geq 0.$$

Thus W_m has a Gamma(m, ν) density.

7. **Optional bonus problem 2:** Let $\Omega = \mathbb{Z}$ =the integers, and let $\mathcal{A}$ be the collection of subsets A of $\mathbb{Z}$ so that A or A^c is finite. Let $\mu(A) = 0$ in the first case, and let $\mu(A) = 1$ in the second. Show that μ has no extension to $\sigma(\mathcal{A})$.