**Statistics 521, Problem Set 10**

Wellner; 11/27/2019

**Reading:** Shorack, PfS, Chapter 8, pages 147 - 174.

**Reminder:** Final exam, 2:30 - 4:20, Wednesday, December 11.

**Due:** Wednesday, December 4, 2019.

1. Let $X_1, X_2, \ldots$ be independent with $P(X_n = 1) = p_n$ and $P(X_n = 0) = 1 - p_n$. Show that: (i) $X_n \to_p 0$ if and only if $p_n \to 0$, and $X_n \to_{a.s.} 0$ if and only if $\sum_n p_n < \infty$.

2. Exercise 8.4.13, PfS page 168: If $X_1, X_2, \ldots$ are i.i.d. Exponential(1) , then $\limsup_n X_n / \log n = 1$ a.s. and $X_{n:n}/\log n \to 1$ a.s.

3. Exercise 8.4.14, PfS page 168: If $X_1, X_2, \ldots$ are i.i.d. Normal$(0, 1)$, then $X_{n:n}/\sqrt{2 \log n} \to 1$ a.s. Hint: You may use the following inequalities for the tail probability of a standard normal distribution:

$$\frac{z}{1+z^2}\phi(z) \le P(Z \ge z) < \frac{1}{z}\phi(z)$$

   where $\phi(z) = (2\pi)^{-1/2} e^{-z^2/2}$.

4. (Monte Carlo Estimation) Exercise 8.5.2, PfS page 174: Let $h : [0, 1] \to [0, 1]$ be continuous.
   (i) Let $X_k \equiv 1_{[h(\xi_k) \ge \Theta_k]}$, where $\xi_1, \xi_1, \ldots$ and $\Theta_1, \Theta_2, \ldots$, are i.i.d Uniform$[0, 1]$ random variables. Show that $\overline{X}_n \to_{a.s.} \int_0^1 h(t)dt$.
   (ii) Let $Y_k \equiv h(\xi_k)$. Show that $\overline{Y}_n \to_{a.s.} \int_0^1 h(t)dt$.
   (iii) Evaluate $Var(\overline{X}_n)$ and $Var(\overline{Y}_n)$ and compare them.

5. **Bonus problem:** Show that $P(Z \ge z) \le (1/2)e^{-z^2/2}$. When is this bound better than the upper bound in the hint for problem 3?