Statistics 521, Practice Midterm Exam B

Wellner; 10/16/2019

1. (24 points). **Define** *three* of the following five terms:
   (a) Convergence in measure of a sequence of measurable functions $\{X_n\}$ defined on the measure space $(\Omega, \mathcal{A}, \mu)$.
   (b) Convergence in distribution of a sequence of random variables $\{X_n\}$.
   (c) $\limsup A_n$ for a sequence of events $\{A_n\}$, and give the common (intuitive) abbreviation for this set.
   (d) A *simple function* defined on a measurable space $(\Omega, \mathcal{A})$.
   (e) The Lebesgue integral $\int X d\mu$ of a (real-valued) measurable function X defined on a measure space $(\Omega, \mathcal{A}, \mu)$.

2. (20 points). Give careful **statements** of *two* of the following five theorems:
   (a) The dominated convergence theorem.
   (b) A theorem relating convergence in measure ($\rightarrow_\mu$) to convergence almost everywhere ($\rightarrow_{a.e.}$).
   (c) The C_r-inequality.
   (d) The Caratheodory extension theorem.
   (e) The "theorem of the unconscious statistician".

3. (30 points).
   (a) Suppose that X is a non-negative measurable function on a measurable space $(\Omega, \mathcal{A})$. Give an explicit sequence of simple functions X_n satisfying $X_n \nearrow X$.
   (b) Now suppose that $(\Omega, \mathcal{A}) = ((0,1), \mathcal{B}_{(0,1)})$, and that we give this measurable space the Lebesgue measure λ, which we call P since it is a probability measure on this $(\Omega, \mathcal{A})$. Suppose that $X(\omega) = \omega^{-2/3}$ for $\omega \in (0,1)$.
   (b-1) For the simple functions X_n as given in (a), evaluate
   $$\lim_{n\to\infty} \int X_n dP = \lim_{n\to\infty} E(X_n).$$
   (b-2) Find the (induced) distribution function $F = F_X$ of X on $\mathbb{R}$.

4. (32 points).
   Let $(\Omega, \mathcal{A}, \mu) = ([0,1], \mathcal{B}, P)$ with $P = \lambda$ being Lebesgue measure (on $[0,1]$). For $\omega \in [0,1]$, define measurable functions X_n by $X_n(\omega) = \omega 1_{(n^{-2},1]} + n 1_{[0,n^{-2}]}$.
   (a) Does $X_n \rightarrow_{a.s.}$ "some" X?
   (b) Does $X_n \rightarrow_p$ "some" X?
   (c) Does $X_n \rightarrow_d$ "some" X? If so, what is the distribution function F of X?
   (d) Let $g(x) = (1/2)x1_{[0,1/2)}(x) + ((1/2)x + 1/2)1_{[1/2,1]}(x)$.
   Does $g(X_n) \rightarrow_p$ something? If so, what is something?
   (e) Compute EX_n^r for $r > 0$ and $n = 1, 2, \ldots$.
   (f) For what values of $r > 0$ does $EX_n^r \rightarrow$ something finite?
   (g) For what values of $r > 0$ does $EX_n^r \rightarrow EX^r$?