Statistics 521, Midterm Exam
Wellner; 11/01/2019

This exam is to be taken without any books or notes.

1. (24 points). **Define** *three* of the following five terms:
   (a) Convergence in probability of a sequence of random variables $\{X_n\}$ defined on a probability space $(\Omega, \mathcal{A}, P)$.
   (b) A measurable function $X : \Omega \to \Omega'$ where $(\Omega, \mathcal{A})$ and $(\Omega', \mathcal{A}')$ are measurable spaces.
   (c) The induced distribution P_X of a random variable $X : (\Omega, \mathcal{A}) \to (\mathbb{R}, \mathcal{B})$.
   (d) A *simple function* defined on a measurable space $(\Omega, \mathcal{A})$.
   (e) Given a measurable space $(\Omega, \mathcal{A})$ define a $\lambda-$system of subsets of Ω and a $\pi-$system of subsets of Ω.

2. (24 points). Give careful **statements** of *two* of the following five theorems or results:
   (a) The monotone convergence theorem.
   (b) A theorem relating Lebesgue-Stieltjes measures μ to (generalized) distribution functions on $\mathbb{R}$
   (c) A result connecting a $\pi-$system $\mathcal{C}$ and a $\lambda-$system $\mathcal{D}$ where $\mathcal{C}$ and $\mathcal{D}$ are collections of subsets of some set Ω.
   (d) Hölder's inequality.
   (e) The Caratheodory extension theorem.

3. (30 points).
   Show that if $X_n \to_p X$ then $X_n \to_d X$.

   Do **either** problem 4 **or** problem 5.

4. (36 points).
   (a) State Jensen's inequality.
   (b) Use Jensen's inequality to prove that $\left(\prod_{i=1}^n x_i\right)^{1/n} \le \{x_1 + \cdots + x_n\}/n$ for any real numbers $x_i \ge 0$ for $i \in \{1, \ldots, n\}$. (Be careful about the case with some $x_j = 0$.) When does equality occur?
   (c) Use Jensen's inequality to prove that $\exp E(\log X) \le E(X)$ for a non-negative random variable X, assuming that $E(\log X)^+ = E(1_{[X \ge 1]} \log X) < \infty$. When does equality occur?

5. (36 points).
   (a) State Markov's inequality.
   (b) Use Markov's inequality or a special case to give a bound for $P(|\overline{X}_n - \mu| \ge t)$ when $X_1, \ldots, X_n$ are independent and identically distributed with $E(X_i) = \mu$ and $Var(X_i) = \sigma^2 < \infty$ for all i.
   (c) Does the bound in (b) show that $\overline{X}_n \to_p \mu$ under these assumptions?

Do **either** problem 6 **or** problem 7.

6. (30 points).
(a) Give an example of a sequence of measurable functions (or random variables) $\{X_n\}$ defined on a probability space $(\Omega, \mathcal{A}, P)$ (which you should make explicit) for which $X_n \to_{a.s.} 0$ but $E(X_n) \to 0, 1,$ or $+\infty$ depending on the value of some number $c > 0$.
(b) Give an example of a sequence of measurable functions (or random variables) $\{X_n\}$ defined on a probability space $(\Omega, \mathcal{A}, P)$ (which you should make explicit) for which $X_n \to_p 0$, but $X_n \not\to_{a.s.} 0$.

7. (30 points).
(a) Suppose that X is a non-negative measurable function on a measurable space $(\Omega, \mathcal{A})$. Give an explicit sequence of simple functions X_n satisfying $X_n \nearrow X$.
(b) Now suppose that $(\Omega, \mathcal{A}) = ((0,1), \mathcal{B}_{(0,1)})$, and that we give this measurable space the Lebesgue measure λ, which we call P since it is a probability measure on this $(\Omega, \mathcal{A}, P)$. Suppose that $X(\omega) = -\log(\omega)$ for $\omega \in (0,1)$.
For the simple functions X_n as given in (a), evaluate

$$\lim_{n\to\infty} \int X_n dP = \lim_{n\to\infty} E(X_n).$$

(c) Find the (induced) distribution function $F = F_X$ of X on $\mathbb{R}$.