Statistics 521, Problem Set 9 Solutions

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1. PfS, Exercise 7.1.1, page 124:
(a) Show that $P(AB) = P(A)P(B)$ if and only if $\{\emptyset, A, A^c, \Omega\}$ and $\{\emptyset, B, B^c, \Omega\}$ are independent $\sigma-$fields.
(b) Show that $A_1, \ldots, A_n$ are independent if and only if (for each $k = 1, \ldots, n$,

$$P(A_{i_1} \ldots A_{i_k}) = \prod_{j=1}^{k} P(A_{i_j}) \qquad \text{whenever } \ 1 \le i_1 < \cdots < i_k \le n\,.$$

Solution: We will prove (b) first.
(b) Since $A_1, \ldots, A_n$ are independent if and only if the random variables $1_{A_1}, \ldots, 1_{A_n}$ are independent, if and only if the $\sigma-$fields $\mathcal{F}(1_{A_1}), \ldots, \mathcal{F}(1_{A_n})$ are independent, and of course these are just the $\sigma-$fields
$\mathcal{A}_1 \equiv \{\emptyset, A_1, A_1^c, \Omega\}, \ldots, \mathcal{A}_n \equiv \{\emptyset, A_n, A_n^c, \Omega\}$.
It remains only to show that the $\sigma-$fields $\mathcal{A}_1, \ldots, \mathcal{A}_n$ are independent if and only if for each $k = 1, \ldots, n$

$$P(A_{i_1} \cap \ldots \cap A_{i_k}) = \prod_{j=1}^{k} P(A_{i_j}) \tag{1}$$

whenever $1 \le i_1 < \cdots < i_k \le n$. First suppose that the $\sigma-$fields $\mathcal{A}_1, \ldots, \mathcal{A}_n$ are independent. Then by taking $\Omega \in \{\emptyset, A_j, A_j^c, \Omega\}$ for $j \in \{i_1, \ldots, i_k\}^c$, we have, with $B_j = A_{i_m}$ if $j = i_m$, $B_j = \Omega$ if $j \in \{i_1, \ldots, i_k\}^c$,

$$P(A_{i_1} \cap \cdots \cap A_{i_k}) = P(\cap_{j=1}^{n} B_j) = \prod_{j=1}^{n} P(B_j) = P(A_{i_1}) \cdots P(A_{i_k})$$

since $P(B_j) = P(\Omega) = 1$ for $j \in \{i_1, \ldots, i_k\}^c$; i.e. (1) holds. Now suppose that (1) holds. In particular, this implies that

$$P(B_1 \cap \ldots \cap B_n) = P(B_1) \cdots P(B_n) \tag{2}$$

where each $B_i \in \{\phi, A_i, \Omega\}$, $i = 1, \ldots, n$, since both sides are 0 if any $B_j = \emptyset$, and if $B_j = A_j$ for $j \in \{j_1, \ldots, j_m\} \subset \{1, \ldots, n\}$, $B_j = \Omega$ for $j \in \{j_1, \ldots, j_m\}^c \subset \{1, \ldots, n\}$, then (2) reduces to (1). Now consider replacing B_1 by the other remaining element of $\mathcal{A}_1$, A_1^c: if we replace B_1 by A_1^c, then since $\Omega = A_1 + A_1^c$ it follows that $\Omega \cap \cap_{k=2}^n A_k = \cap_{k=1}^n A_k + A_1^c \cap_{k=2}^n A_k$ and hence the left side of (2) becomes

$$\begin{aligned} P(\cap_{k=2}^n B_k) - P(A_1 \cap \cap_{k=2}^n B_k) &= P(B_2) \cdots P(B_n) - P(A_1)P(B_2) \cdots P(B_n) \\ &= (1 - P(A_1))P(B_2) \cdots P(B_n) \\ &= P(A_1^c)P(B_2) \cdots P(B_n)\,; \end{aligned}$$

thus we have proved that

$$P(C_1 \cap B_2 \ldots \cap B_n) = P(C_1)P(B_2) \cdots P(B_n) \tag{3}$$

for $C_1 \in \mathcal{A}_1$, and $B_j \in \{\emptyset, A_j, \Omega\}$ for $j = 2, \ldots, n$. This is the first step of an induction argument. Now suppose that for some $k \in \{1, \ldots, n\}$.

$$P(C_1 \cdots C_{k-1} \cap B_k \cdots B_n) = P(C_1) \cdots P(C_{k-1})P(B_k) \cdots P(B_n) \tag{4}$$

for all $C_i \in \mathcal{A}_i$, $i = 1, \ldots, k-1$, $B_i \in \{\emptyset, A_i, \Omega\}$, $i = k, \ldots, n$. Since $\Omega = A_k + A_k^c$ it follows that

$$\Omega \cap_{j=1}^{k-1} C_j \cap_{j=k+1}^n B_j = A_k \cap_{j=1}^{k-1} C_j \cap_{j=k+1}^n B_j + A_k^c \cap_{j=1}^{k-1} C_j \cap_{j=k+1}^n B_j\,.$$

Thus upon replacing B_k by A_k^c on the left side of (4), we see that we have, since both $\Omega, A_k \in \{\emptyset, A_k, \Omega\}$,

$$\begin{aligned} &P(\cap_{j=1}^{k-1} C_j \cap A_k^c \cap \cap_{j=k+1}^n B_j) \\ &= P(\cap_{j=1}^{k-1} C_j \cap \Omega \cap \cap_{j=k+1}^n B_j) - P(\cap_{j=1}^{k-1} C_j \cap A_k \cap \cap \cap_{j=k+1}^n B_j) \\ &= \prod_{j=1}^{k-1} P(C_j)P(\Omega) \prod_{j=k+1}^n P(B_j) - \prod_{j=1}^{k-1} P(C_j)P(A_k) \prod_{j=k+1}^n P(B_j) \\ &= \prod_{j=1}^{k-1} P(C_j) \cdot (1 - P(A_k)) \cdot \prod_{j=k+1}^n P(B_j) \\ &= \prod_{j=1}^{k-1} P(C_j) \cdot P(A_k^c) \cdot \prod_{j=k+1}^n P(B_j)\,. \end{aligned}$$

Hence we have proved that (4) implies that

$$P(C_1 \cdots C_k B_{k+1} \cdots B_n) = P(C_1) \cdots P(C_k) P(B_{k+1}) \cdots P(B_n) \qquad (5)$$

for all $C_i \in \mathcal{A}_i$, $i = 1, \ldots, k$, $B_i \in \{\emptyset, A_i, \Omega\}$, $i = k+1, \ldots, n$. It then follows by induction that

$$P(C_1 \cdots C_n) = P(C_1) \cdots P(C_n) \qquad (6)$$

for all $C_i \in \mathcal{A}_i$, $i = 1, \ldots, n$; i.e. the $\sigma-$fields $\mathcal{A}_i$, $i = 1, \ldots, n$ are independent.

(a) This follows immediately from (a) with $n = 2$.

2. Give an example of two collections of sets $\mathcal{A}_1$ and $\mathcal{A}_2$ that are independent but the generated $\sigma-$fields are not independent.

   Solution: One example of this goes as follows: let $\Omega = \{1, 2, 3, 4\}$, and let $\mathcal{A} = 2^\Omega$. Suppose that $P(\{\omega\}) = 1/4$ for each $\omega \in \Omega$. Suppose that $\mathcal{A}_1 = \{\{1, 2\}\}$ and $\mathcal{A}_2 = \{\{2, 3\}, \{2, 4\}\}$. For $A_1 \in \mathcal{A}_1$, $A_2 \in \mathcal{A}_2$, we have

   $$P(A_1 \cap A_2) = P(\{2\}) = 1/4 = (1/2)(1/2) = P(A_1)P(A_2)$$

   so that $\mathcal{A}_1$, $\mathcal{A}_2$ are independent classes. Then we have $\{2\} \in \sigma[\mathcal{A}_2]$ (since $\{2\} = \{2, 3\} \cap \{2, 4\}$) and $\{1, 2\} \in \sigma[\mathcal{A}_1]$, but

   $$P(\{1, 2\} \cap \{2\}) = P(\{2\}) = 1/4 \neq 1/8 = P(\{1, 2\})P(\{2\}) .$$

   Thus $\sigma[\mathcal{A}_1]$ and $\sigma[\mathcal{A}_2]$ are *not* independent classes. The difficulty here is that the class $\mathcal{A}_2$ is not a $\overline{\pi}-$system.

3. Show that if X_n is any sequence of random variables, there are constants $c_n \to \infty$ so that $X_n / c_n \to_{a.s.} 0$.

   Solution: Define $F_n(x) \equiv P(|X_n| \leq x)$, the distribution function of $|X_n|$. Set $b_n = F_n^{-1}(1 - n^{-2})$ for $n = 1, 2, \ldots$, and let $\{a_n\}$ be any sequence with $a_n \to \infty$. Let $c_n = a_n b_n$. Then, for any $\epsilon > 0$ we have $\epsilon a_n \geq 1$ for $n \geq N_\epsilon$ and hence, using $F_n \circ F_n^{-1}(t) \geq t$ for all $0 < t < 1$,

   $$\begin{aligned} P(|X_n| > \epsilon c_n) &= P(|X_n| > \epsilon a_n b_n) \\ &\leq P(|X_n| > b_n) \\ &= 1 - F_n(b_n) = 1 - F_n(F_n^{-1}(1 - n^{-2})) \\ &\leq n^{-2}, \quad n \geq N_\epsilon. \end{aligned}$$

Hence by the first Borel-Cantelli lemma, $P(|X_n| > \epsilon c_n \;\text{ i.o.}) = 0$ for every $\epsilon > 0$; that is, $X_n/c_n \to_{a.s.} 0$.

4. Show that if $P(A_n) \to 0$ and $\sum_{n=1}^{\infty} P(A_n \cap A_{n+1}^c) < \infty$, then $P(A_n \text{i.o.}) = 0$.

Solution 1: Let $B_n \equiv A_n \cap A_{n+1}^c$. Then by the first Borel-Cantelli lemma, $P(\limsup_n B_n) = P(B_n\ i.o.) = 0$. Furthermore, since $P(A_n) \to 0$ it follows that

$$P(\limsup_n A_n^c) = \lim_{n\to\infty} P(\cup_{m\geq n} A_m^c) \geq \lim_{\to\infty} P(A_n^c) = 1.$$

Suppose that the following claim holds:
Claim: $\limsup_n A_n \cap \limsup_n A_n^c \subset B_n$.
Then

$$\begin{aligned} P(\limsup_n A_n) &= P(\limsup_n A_n \cap \limsup_n A_n^c) + P(\limsup_n A_n \cap [\limsup_n A_n^c]^c) \\ &\leq P(\limsup B_n) + P([\limsup A)_n^c]^c) = 0 + 0 = 0. \end{aligned}$$

To prove the claim, let $\omega \in \limsup_n A_n \cap (\limsup_n A_n^c)$. Fix $N \geq 1$. Then there exists an $m \geq N$ such that $\omega \in A_m$. Now let $t \;\inf\{k > m : \omega \in A_k^c\}$. Such a t exists since $\omega \in \limsup_n A_n^c$. But then $\omega \in A_{t-1} \cap A_t^c$, and hence

$$\omega \in \cap_{n=1}^{N} \cup_{m=n}^{\infty} (A_m \cap A_{m+1}^c).$$

Since this holds for any N we have $\omega \in \cap_{n=1}^{\infty} \cup_{m=n}^{\infty} (A_m \cap A_{m+1}^c) = \limsup_n B_n$.

Solution 2: First we write

$$[A_n\ i.o.] = \lim_n \cup_{m=n}^{\infty} A_m = \lim_n \lim N \cup_{m=n}^{N} A_m$$

and hence

$$P[A_n\ i.o.] = \lim_n \lim N P(\cup_{m=n}^{N} A_m).$$

Now write

$$\cup_{m=n}^{N} A_m = \left(\cup_{m=n}^{N-1} (A_m \cap A_{m+1}^c)\right) \cup A_N,$$

and hence

$$P(\cup_{m=n}^{N} A_m) \leq P\left(\cup_{m=n}^{N-1}(A_m \cap A_{m+1}^c)\right) + P(A_N).$$

This yields

$$\begin{aligned}
P[A_n \ i.o.] &= \lim_n \lim N P(\cup_{m=n}^{N} A_m) \\
&\leq \lim_n \lim N \left\{P\left(\cup_{m=n}^{N-1}(A_m \cap A_{m+1}^c)\right) + P(A_N)\right\} \\
&\leq \lim_n \sum_{m=n}^{\infty} P(A_m \cap A_{m+1}^c) + \lim_{N\to\infty} P(A_N) \\
&= 0 + 0 = 0.
\end{aligned}$$

5. Let $X_1, X_2, \ldots$ be independent. Show that $\sup_{n\geq 1} X_n < \infty$ almost surely if and only if $\sum_{n=1}^{\infty} P(X_n > M) < \infty$ for some $M < \infty$.

   Solution: Suppose that $\sum_n P(X_n > M) < \infty$ for some $M < \infty$. Then by the first Borel-Cantelli lemma, $P(X_n > M \text{ i.o.}) = 0$; i.e. for $n \geq N_\omega$ we have $X_n(\omega) \leq M$. Thus

   $$\sup_n X_n(\omega) \leq \left(\max_{1\leq k<N_\omega} X_k\right) \vee M \ < \ \infty\,.$$

   Now suppose that $\sup X_n < \infty$ almost surely. If $\sum_n P(X_n > M) = \infty$ for every $M < \infty$, then, by the second Borel-Cantelli lemma, $P(X_n > M \text{ i.o.}) = 1$ for every M; i.e. $\limsup_{n\to\infty} X_n \geq M$ a.s. for every $M > 0$, and this implies, by taking a sequence $M_k \nearrow \infty$, that $\limsup_{n\to\infty} X_n = \infty$ a.s., which contradicts $\sup X_n < \infty$ almost surely. We therefore conclude that $\sum_n P(X_n > M) < \infty$ for some $M < \infty$.

6. Suppose that $X_1, X_2, \ldots$ are independent with $P(X_n > x) = x^{-r}$ for all $x \geq 1$ and $n = 1, 2, \ldots$ with $r > 0$. Show that $\limsup_{n\to\infty}(\log X_n)/\log n = c$ almost surely for some number c, and find c.

   Solution: Let $c > 0$. Then

   $$P(\log X_n > c \log n) = P(X_n > n^c) = n^{-5r}.$$

   Hence by the first and second Borel-Cantelli lemmas

   $$P(\log X_n > c\log n \ \text{ i.o.}) = \begin{cases} 0 & \text{if } c > 1/r \\ 1 & \text{if } c \leq 1/r \end{cases}\,.$$

Hence

$$\limsup_{n\to\infty}(\log X_n)/(\log n) = 1/r \qquad \text{almost surely}\,.$$