

## Statistics 521, Problem Set 8 Solutions

Wellner; 11/30/2016

1. PfS Exercise 6.4.3, page 114. Prove just the parts of these formulas involving  $F$ , not the parts involving  $F^{-1}$ . You may also use Fubini's theorem directly. That is, show that:
  - (i) If  $X \geq 0$  has d.f.  $F$ , then
$$\int_0^\infty P(X > x)dx = E(X) = \int_0^\infty (1 - F(x))dx.$$
  - (ii) If  $E|X| < \infty$  then
$$E(X) = -\int_{-\infty}^0 F(x)dx + \int_0^\infty (1 - F(x))dx.$$
  - (iii) Let  $r > 0$ . If  $X \geq 0$ , then
$$\int_0^\infty P(X^r > x)dx = E(X^r) = \int_0^\infty rx^{r-1}(1 - F(x))dx.$$

**Solution:** (11): Since  $X \geq 0$  we may write  $X = \int_0^X dt$  to obtain

$$\begin{aligned} E(X) &= \int_{\Omega} X dP = \int_{\Omega} \int_0^X dt dP \\ &= \int_{\Omega} \int_0^\infty 1_{\{0 \leq t < X\}} dt dP \\ &= \int_0^\infty \int_{\Omega} 1_{\{0 \leq t < X\}} dP dt && \text{by Fubini's theorem} \\ &= \int_0^\infty P(X > t) dt = \int_0^\infty (1 - F(t)) dt. \end{aligned}$$

If either side is  $+\infty$ , then both sides are  $+\infty$ .

(12): Since  $X = X^+ - X^-$  with  $X^+, X^- \geq 0$ , it follows from (11) applied to  $X^+$  and  $X^-$  (and assuming that  $E|X| = EX^+ + EX^- < \infty$ )

that

$$\begin{aligned}
E(X) &= E(X^+) - E(X^-) \\
&= \int_0^\infty P(X^+ > t)dt - \int_0^\infty P(X^- > t)dt \\
&= \int_0^\infty P(X > t)dt - \int_0^\infty P(-X > t)dt \\
&= \int_0^\infty (1 - F(t))dt - \int_0^\infty P(X < -t)dt \\
&= \int_0^\infty (1 - F(t))dt - \int_0^\infty F(-t-)dt \\
&= \int_0^\infty (1 - F(t))dt - \int_{-\infty}^0 F(t-)dt \\
&= \int_0^\infty (1 - F(t))dt - \int_{-\infty}^0 F(t)dt
\end{aligned}$$

where the last equality follows by a change of variables and the fact that  $F(t)$  differs from  $F(t-)$  on a set which has Lebesgue measure at most zero.

(13): Since  $X \geq 0$  we may write  $X^r = \int_0^X rt^{r-1}dt$  to obtain

$$\begin{aligned}
E(X^r) &= \int_\Omega X dP = \int_\Omega \int_0^X rt^{r-1}dt dP = \int_\Omega \int_0^\infty 1_{\{0 \leq t < X\}} rt^{r-1}dt dP \\
&= \int_0^\infty \int_\Omega 1_{\{0 \leq t < X\}} dP rt^{r-1}dt \quad \text{by Fubini's theorem} \\
&= \int_0^\infty rt^{r-1}P(X > t)dt = \int_0^\infty rt^{r-1}(1 - F(t))dt.
\end{aligned}$$

If either side is  $+\infty$ , then both sides are  $+\infty$ .

(14): First suppose that  $X$  and  $Y$  are both non-negative and that  $G$  and  $H$  satisfy  $G_-(0) = G_+(0) = H_-(0) = H_+(0) = 0$ . Then we can write  $G(x) = \int_{[0,x)} 1_{[0,x)}(s)dG_-(s)$ ,  $H(y) = \int_{[0,y)} 1_{[0,y)}(t)dH_-(t)$ . Now finiteness of the covariance  $Cov[G(X), H(Y)]$  implies that  $(G(X) - EG(X))(H(Y) - EH(Y))$  is integrable and hence that  $G(X)H(Y)$  is integrable. Furthermore

$$\begin{aligned}
Cov[G(X), H(Y)] &= E\{[G(X) - EG(X)][H(Y) - EH(Y)]\} \\
&= E\{G(X)H(Y)\} - EG(X) \cdot EH(Y).
\end{aligned}$$

Thus Fubini's theorem yields

$$\begin{aligned}
E\{G(X)H(Y)\} &= \int_{[0,\infty)} \int_{[0,\infty)} G(x)H(y)dF(x,y) \\
&= \int_{[0,\infty)} \int_{[0,\infty)} \left\{ \int_{[0,\infty)} 1_{[0,x)}(s)dG_-(s) \int_{[0,\infty)} 1_{[0,y)}(t)dH_-(t) \right\} dF(x,y) \\
&= \int_{[0,\infty)} \int_{[0,\infty)} \left\{ \int_{[0,\infty)} \int_{[0,\infty)} 1_{[0,x)}(s)1_{[0,y)}(t)dF(x,y) \right\} dG(s)dH(t) \\
&\quad \text{by Fubini's theorem} \\
&= \int_{[0,\infty)} \int_{[0,\infty)} P(X > s, Y > t)dG(s)dH(t);
\end{aligned}$$

note that this is always finite. Furthermore

$$\begin{aligned}
EG(X) &= \int_{[0,\infty)} G(x)dF_X(x) = \int_{[0,\infty)} \int_{[0,\infty)} 1_{[0,x)}(s)dG_-(s)dF_X(x) \\
&= \int_{[0,\infty)} \left\{ \int_{[0,\infty)} 1_{[0,x)}(s)dF_X(x) \right\} dG(s) \\
&= \int_{[0,\infty)} P(X > s)dG(s) < \infty
\end{aligned}$$

and, similarly,

$$EH(Y) = \int_{[0,\infty)} P(Y > t)dH(t) < \infty.$$

Combining these yields

$$\begin{aligned}
&E\{G(X)H(Y)\} - EG(X) \cdot EH(Y) \\
&= \int_0^\infty \int_0^\infty \{P(X > s, Y > t) - P(X > s)P(Y > t)\}dG(s)dH(t) \\
&= \int_0^\infty \int_0^\infty \{F(s,t) - F_X(s)F_Y(t)\}dG(s)dH(t)
\end{aligned}$$

using the relations

$$\begin{aligned}
P(X > s, Y > t) &= 1 - F_X(s) - F_Y(t) + F(s,t), \\
P(X > s)P(Y > t) &= (1 - F_X(s))(1 - F_Y(t)) \\
&= 1 - F_X(s) - F_Y(t) + F_X(s)F_Y(t).
\end{aligned}$$

For general  $X, Y$  we can repeat this argument for each of the four quadrants, with each quadrant contributing a finite term, and add the results to obtain the claimed formula.

2. Prove the two formulas in (17), PFS page 113: if  $X \geq 0$  is integer valued, then  $E(X) = \sum_{k=1}^{\infty} P(X \geq k)$  and  $E(X^2) = \sum_{k=1}^{\infty} (2k-1)P(X \geq k)$ .

**Solution:** First note that if  $X \geq 0$  is integer-valued, then the distribution function  $F$  of  $X$  is constant between integers, and  $P(X > x) = 1 - F(x) = P(X > k)$  for  $k \leq x < k+1$ . Thus from (7.4.11) we find that

$$\begin{aligned} E(X) &= \int_0^{\infty} (1 - F(x)) dx \\ &= \sum_{k=0}^{\infty} \int_{[k, k+1)} (1 - F(x)) dx \\ &= \sum_{k=0}^{\infty} (1 - F(k)) \int_{[k, k+1)} dx \\ &= \sum_{k=0}^{\infty} P(X > k) = \sum_{k=0}^{\infty} P(X \geq k+1) \\ &= \sum_{m=1}^{\infty} P(X \geq m). \end{aligned}$$

Similarly, from (7.4.13) with  $r = 2$ ,

$$\begin{aligned} E(X^2) &= \int_0^{\infty} 2x(1 - F(x)) dx \\ &= \sum_{k=0}^{\infty} \int_{[k, k+1)} 2x(1 - F(x)) dx \\ &= \sum_{k=0}^{\infty} P(X > k) \int_{[k, k+1)} 2x dx \\ &= \sum_{k=0}^{\infty} P(X > k)(2k+1) \end{aligned}$$

since

$$\int_k^{k+1} 2x dx = x^2 \Big|_k^{k+1} = 2k+1.$$

3. PfS Exercise 4.9, page 114: For any distribution function  $F$  on  $\mathbb{R}$  we have  $\int \{F(x + \theta) - F(x)\} dx = \theta$  for each  $\theta \geq 0$ .

**Solution:** First note that

$$F(x + \theta) - F(x) = \int_{(x, x+\theta]} dF(y) = \int_{-\infty}^{\infty} 1_{(x, x+\theta]}(y) dF(y)$$

where the integrand,  $h(x, y) \equiv 1_{(x, x+\theta]}(y) \geq 0$  for all  $x, y$ . Thus by the Tonelli corollary of Fubini's theorem

$$\begin{aligned} \int \{F(x + \theta) - F(x)\} dx &= \int \left\{ \int_{-\infty}^{\infty} 1_{(x, x+\theta]}(y) dF(y) \right\} dx \\ &= \int_{-\infty}^{\infty} \left\{ \int 1_{(x, x+\theta]}(y) dx \right\} dF(y) \\ &= \int_{-\infty}^{\infty} \left\{ \int 1_{[y-\theta, y)}(x) dx \right\} dF(y) \\ &= \int_{-\infty}^{\infty} \{\theta\} dF(y) = \theta \cdot 1 = \theta. \end{aligned}$$

4. PfS Exercise 4.11, page 114:

(a) Show that  $\int_0^{\infty} \{P(|X| > x)\}^{1/2} dx < \infty$  implies  $E(X^2) < \infty$ .

(b) Show that  $\int_0^{\infty} \{P(|X| > x)\}^{1/2} dx \leq \frac{r}{r-2} \|X\|_r$  for any  $r > 2$  so that the integral on the left is finite whenever  $X \in \mathcal{L}_r$  for any  $r > 2$ . If  $\int_0^{\infty} \{P(|X| > x)\}^{1/2} dx < \infty$  then we say that  $X \in \mathcal{L}_{2,1}$ ; this condition arises in connection with optimal transportation inequalities for empirical processes and in multiplier and bootstrap CLTs.

**Solution:** (a) Now  $t^2 P(|X| \geq t) \leq E\{|X|^2 1_{\{|X| \geq t\}}\} \leq E|X|^2$  for all  $t \geq 0$ . Thus, using problem 1 and letting  $t_0 > 0$ ,

$$\begin{aligned} E|X|^2 &= \int_0^{\infty} 2tP(|X| > t) dt = \int_0^{t_0} 2tP(|X| > t) dt + \int_{t_0}^{\infty} 2tP(|X| > t) dt \\ &\leq \int_0^{t_0} 2t dt + \int_{t_0}^{\infty} 2\sqrt{t^2 P(|X| > t)} \sqrt{P(|X| > t)} dt \\ &\leq t_0^2 + 2\|X\|_2 \int_{t_0}^{\infty} \sqrt{P(|X| > t)} dt \\ &\leq 2^{-1}\|X\|_2^2 + 2\|X\|_2 \int_0^{\infty} \sqrt{P(|X| > t)} dt \text{ by choosing } t_0 = 2^{-1/2}\|X\|_2. \end{aligned}$$

Rearranging terms yields

$$\frac{1}{2}\|X\|_2^2 \leq 2\|X\|_2 \int_0^\infty \sqrt{P(|X| > t)} dt,$$

and hence

$$\|X\|_2 \leq 4 \int_0^\infty \sqrt{P(|X| > t)} dt.$$

(b) We proceed somewhat similarly to the proof in (a): for any fixed  $t_0 \in (0, \infty)$  we have, by Markov's inequality,

$$\begin{aligned} \int_0^\infty \sqrt{P(|X| > t)} dt &= \int_0^{t_0} \sqrt{P(|X| > t)} dt + \int_{t_0}^\infty \sqrt{\frac{\|X\|_r^r}{t^r}} dt \\ &\leq t_0 + \|X\|_r^{r/2} \frac{t_0^{1-r/2}}{r/2 - 1}. \end{aligned}$$

The right side in the last line of the last display is minimized by  $t_0 \equiv \|X\|_r$ , and the minimum value is

$$\|X\|_r + \|X\|_r^{r/2} \frac{\|X\|_r^{1-r/2}}{r/2 - 1} = \frac{r}{r-2} \|X\|_r.$$

5. PfS, Exercise 5.1.4, page 91.

Let  $\mathcal{X} = [0, 1]$ ,  $\mathcal{Y} = (1, \infty)$  both equipped with the Borel sets and Lebesgue measure. Let  $f(x, y) = e^{-xy} - 2e^{-2xy}$ . Show that

- (i)  $\int_0^1 (\int_1^\infty f(x, y) dy) dx = \int_0^1 x^{-1} (e^{-x} - e^{-2x}) dx$  exists and is  $> 0$ .
- (ii)  $\int_1^\infty (\int_0^1 f(x, y) dx) dy = \int_1^\infty y^{-1} (e^{-2y} - e^{-y}) dy$  exists and is  $< 0$ .

**Solution:** First,

$$\int_0^1 \left( \int_1^\infty f(x, y) dy \right) dx = \int_0^1 x^{-1} (e^{-x} - e^{-2x}) dx$$

by an easy calculation of the inner integral. Note that the function  $x^{-1}(e^{-x} - e^{-2x})$  converges to 1 as  $x \downarrow 0$ , and is continuous elsewhere, hence is uniformly continuous and uniformly bounded on  $[0, 1]$ . Since it is strictly positive and bounded, the last integral exists and is positive. On the other hand,

$$\int_1^\infty \left( \int_0^1 f(x, y) dx \right) dy = \int_1^\infty y^{-1} (e^{-2y} - e^{-y}) dy$$

again by an easy calculation of the inner integral. Now the integrand is negative (since  $e^{-2y} < e^{-y}$  for all  $y > 0$ ), bounded and the two integrals  $\int_1^\infty y^{-1}e^{-2y}dy$  and  $\int_1^\infty y^{-1}e^{-y}dy$  both are clearly finite. Thus the second iterated integral exists and is strictly negative. Letting  $Ei(z) \equiv -\int_z^\infty t^{-1}e^{-t}dt$ , the exponential integral function, it is easily shown that

$$\begin{aligned}\int_0^1 \left( \int_1^\infty f(x, y) dy \right) dx &= \int_0^1 x^{-1} (e^{-x} - e^{-2x}) dx \\ &= -Ei(-2) + Ei(-1) + \log(2) \\ &\doteq 0.522664\dots,\end{aligned}$$

while

$$\begin{aligned}\int_1^\infty \left( \int_0^1 f(x, y) dx \right) dy &= \int_1^\infty y^{-1} (e^{-2y} - e^{-y}) dy \\ &= -Ei(-2) + Ei(-1) \\ &\doteq -0.170483\dots\end{aligned}$$

Of course the difficulty here is that

$$\int_1^\infty \left( \int_0^1 |f(x, y)| dx \right) dy = \int_0^1 \left( \int_1^\infty |f(x, y)| dy \right) dx = \infty.$$

In fact

$$\begin{aligned}\int_0^1 \left( \int_1^\infty f^+(x, y) dx \right) dy &= \int_0^1 \left( \int_{x^{-1} \log 2}^\infty f(x, y) dy \right) dx \\ &= \int_0^1 x^{-1} (e^{-\log 2} - e^{-2 \log 2}) dx \\ &= \frac{1}{4} \int_0^1 x^{-1} dx = +\infty,\end{aligned}$$

and similarly

$$\begin{aligned}\int_0^1 \left( \int_1^\infty f^-(x, y) dx \right) dy &= \int_0^1 \left( \int_0^{x^{-1} \log 2} f(x, y) dy \right) dx \\ &= \int_0^1 x^{-1} ((1 - e^{-2 \log 2}) - (1 - e^{-\log 2})) dx \\ &= \frac{1}{4} \int_0^1 x^{-1} dx = +\infty.\end{aligned}$$

See Figures 1-3.

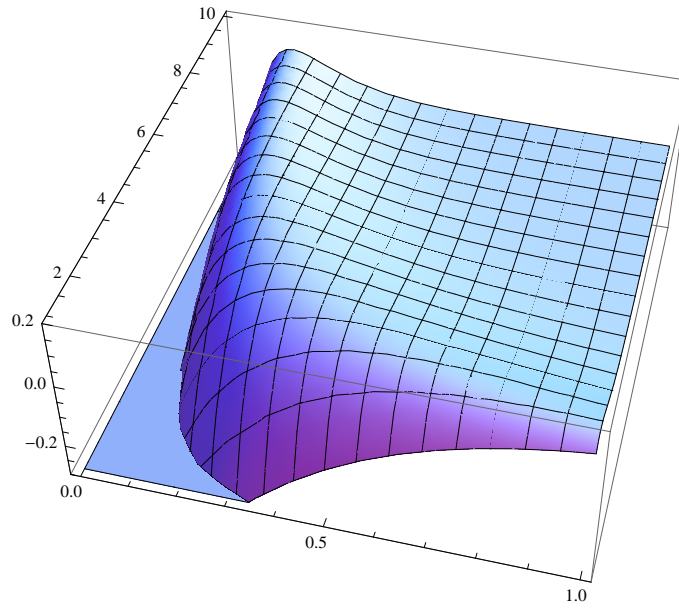


Figure 1: The function  $f(x, y)$ .

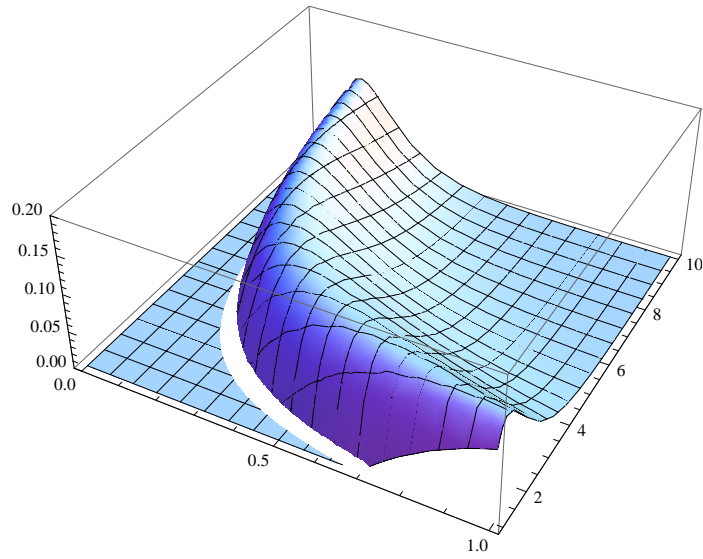


Figure 2: The function  $f^+(x, y)$ .

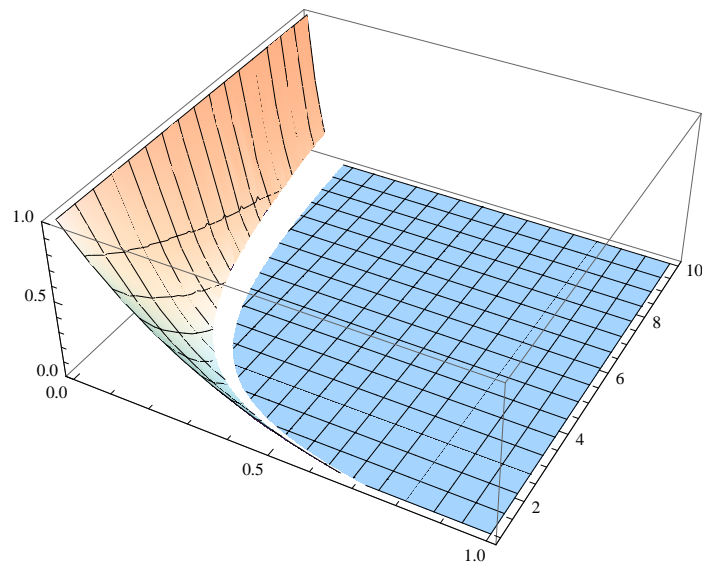


Figure 3: The function  $f^-(x, y)$ .