Statistics 521, Problem Set 8 Solutions

Wellner; 11/30/2016

1. PfS Exercise 6.4.3, page 114. Prove just the parts of these formulas involving F, not the parts involving F^{-1}. You may also use Fubini's theorem directly. That is, show that:
(i) If $X \ge 0$ has d.f. F, then
$\int_0^\infty P(X > x)dx = E(X) = \int_0^\infty (1 - F(x))dx$.
(ii) If $E|X| < \infty$ then
$E(X) = -\int_{-\infty}^0 F(x)dx + \int_0^\infty (1 - F(x))dx$.
(iii) Let $r > 0$. If $X \ge 0$, then
$\int_0^\infty P(X^r > x)dx = E(X^r) = \int_0^\infty rx^{r-1}(1 - F(x))dx$.

Solution: (11): Since $X \ge 0$ we may write $X = \int_0^X dt$ to obtain

$$\begin{aligned} E(X) &= \int_\Omega X dP = \int_\Omega \int_0^X dt dP \\ &= \int_\Omega \int_0^\infty 1_{\{0 \le t < X\}} dt dP \\ &= \int_0^\infty \int_\Omega 1_{\{0 \le t < X\}} dP dt \qquad \text{by Fubini's theorem} \\ &= \int_0^\infty P(X > t)dt = \int_0^\infty (1 - F(t))dt \,. \end{aligned}$$

If either side is $+\infty$, then both sides are $+\infty$.
(12): Since $X = X^+ - X^-$ with $X^+, X^- \ge 0$, it follows from (11) applied to X^+ and X^- (and assuming that $E|X| = EX^+ + EX^- < \infty$)

that

$$\begin{aligned}
E(X) &= E(X^+) - E(X^-) \\
&= \int_0^\infty P(X^+ > t)dt - \int_0^\infty P(X^- > t)dt \\
&= \int_0^\infty P(X > t)dt - \int_0^\infty P(-X > t)dt \\
&= \int_0^\infty (1 - F(t))dt - \int_0^\infty P(X < -t)dt \\
&= \int_0^\infty (1 - F(t))dt - \int_0^\infty F(-t-)dt \\
&= \int_0^\infty (1 - F(t))dt - \int_{-\infty}^0 F(t-)dt \\
&= \int_0^\infty (1 - F(t))dt - \int_{-\infty}^0 F(t)dt
\end{aligned}$$

where the last equality follows by a change of variables and the fact that $F(t)$ differs from $F(t-)$ on a set which has Lebesgue measure at most zero.
(13): Since $X \geq 0$ we may write $X^r = \int_0^X rt^{r-1}dt$ to obtain

$$\begin{aligned}
E(X^r) &= \int_\Omega X dP = \int_\Omega \int_0^X rt^{r-1}dtdP = \int_\Omega \int_0^\infty 1_{\{0 \leq t < X\}} rt^{r-1}dtdP \\
&= \int_0^\infty \int_\Omega 1_{\{0 \leq t < X\}} dP rt^{r-1}dt \qquad \text{by Fubini's theorem} \\
&= \int_0^\infty rt^{r-1} P(X > t)dt = \int_0^\infty rt^{r-1}(1 - F(t))dt\,.
\end{aligned}$$

If either side is $+\infty$, then both sides are $+\infty$.
(14): First suppose that X and Y are both non-negative and that G and H satisfy $G_-(0) = G_+(0) = H_-(0) = H_+(0) = 0$. Then we can write $G(x) = \int_{[0,\infty)} 1_{[0,x)}(s)dG_-(s)$, $H(y) = \int_{[0,\infty)} 1_{[0,y)}(t)dH_-(t)$. Now finiteness of the covariance $Cov[G(X), H(Y)]$ implies that $(G(X) - EG(X))(H(Y) - EH(Y))$ is integrable and hence that $G(X)H(Y)$ is integrable. Furthermore

$$\begin{aligned}
Cov[G(X), H(Y)] &= E\{[G(X) - EG(X)][H(Y) - EH(Y)]\} \\
&= E\{G(X)H(Y)\} - EG(X) \cdot EH(Y).
\end{aligned}$$

Thus Fubini's theorem yields

$$
\begin{aligned}
E\{G(X)H(Y)\} &= \int_{[0,\infty)} \int_{[0,\infty)} G(x)H(y)dF(x,y) \\
&= \int_{[0,\infty)} \int_{[0,\infty)} \left\{ \int_{[0,\infty)} 1_{[0,x)}(s)dG_-(s) \int_{[0,\infty)} 1_{[0,y)}(t)dH_-(t) \right\} dF(x,y) \\
&= \int_{[0,\infty)} \int_{[0,\infty)} \left\{ \int_{[0,\infty)} \int_{[0,\infty)} 1_{[0,x)}(s)1_{[0,y)}(t)dF(x,y) \right\} dG(s)dH(t) \\
&\qquad \text{by Fubini's theorem} \\
&= \int_{[0,\infty)} \int_{[0,\infty)} P(X > s, Y > t)dG(s)dH(t);
\end{aligned}
$$

note that this is always finite. Furthermore

$$
\begin{aligned}
EG(X) &= \int_{[0,\infty)} G(x)dF_X(x) = \int_{[0,\infty)} \int_{[0,\infty)} 1_{[0,x)}(s)dG_-(s)dF_X(x) \\
&= \int_{[0,\infty)} \left\{ \int_{[0,\infty)} 1_{[0,x)}(s)dF_X(x) \right\} dG(s) \\
&= \int_{[0,\infty)} P(X > s)dG(s) < \infty
\end{aligned}
$$

and, similarly,

$$
EH(Y) = \int_{[0,\infty)} P(Y > t)dH(t) < \infty.
$$

Combining these yields

$$
\begin{aligned}
&E\{G(X)H(Y)\} - EG(X) \cdot EH(Y) \\
&\quad = \int_0^\infty \int_0^\infty \{P(X > s, Y > t) - P(X > s)P(Y > t)\}dG(s)dH(t) \\
&\quad = \int_0^\infty \int_0^\infty \{F(s,t) - F_X(s)F_Y(t)\}dG(s)dH(t)
\end{aligned}
$$

using the relations

$$
\begin{aligned}
P(X > s, Y > t) &= 1 - F_X(s) - F_Y(t) + F(s,t), \\
P(X > s)P(Y > t) &= (1 - F_X(s))(1 - F_Y(t)) \\
&= 1 - F_X(s) - F_Y(t) + F_X(s)F_Y(t).
\end{aligned}
$$

For general X, Y we can repeat this argument for each of the four quadrants, with each quadrant contributing a finite term, and add the results to obtain the claimed formula.

2. Prove the two formulas in (17), PfS page 113: if $X \geq 0$ is integer valued, then $E(X) = \sum_{k=1}^{\infty} P(X \geq k)$ and $E(X^2) = \sum_{k=1}^{\infty}(2k-1)P(X \geq k)$.

   Solution: First note that if $X \geq 0$ is integer-valued, then the distribution function F of X is constant between integers, and $P(X > x) = 1 - F(x) = P(X > k)$ for $k \leq x < k+1$. Thus from (7.4.11) we find that

$$\begin{aligned}
E(X) &= \int_0^\infty (1 - F(x))dx \\
&= \sum_{k=0}^{\infty} \int_{[k,k+1)} (1 - F(x))dx \\
&= \sum_{k=0}^{\infty} (1 - F(k)) \int_{[k,k+1)} dx \\
&= \sum_{k=0}^{\infty} P(X > k) = \sum_{k=0}^{\infty} P(X \geq k+1) \\
&= \sum_{m=1}^{\infty} P(X \geq m)\,.
\end{aligned}$$

   Similarly, from (7.4.13) with $r = 2$,

$$\begin{aligned}
E(X^2) &= \int_0^\infty 2x(1 - F(x))dx \\
&= \sum_{k=0}^{\infty} \int_{[k,k+1)} 2x(1 - F(x))dx \\
&= \sum_{k=0}^{\infty} P(X > k) \int_{[k,k+1)} 2xdx \\
&= \sum_{k=0}^{\infty} P(X > k)(2k+1)
\end{aligned}$$

   since

$$\int_k^{k+1} 2xdx = x^2\Big|_k^{k+1} = 2k+1\,.$$

3. PfS Exercise 4.9, page 114: For any distribution function F on $\mathbb{R}$ we have $\int \{F(x+\theta) - F(x)\}dx = \theta$ for each $\theta \geq 0$.

Solution: First note that

$$F(x+\theta) - F(x) = \int_{(x,x+\theta]} dF(y) = \int_{-\infty}^{\infty} 1_{(x,x+\theta]}(y)dF(y)$$

where the integrand, $h(x,y) \equiv 1_{(x,x+\theta]}(y) \geq 0$ for all x, y. Thus by the Tonelli corollary of Fubini's theorem

$$\begin{aligned}
\int \{F(x+\theta) - F(x)\}dx &= \int \left\{ \int_{-\infty}^{\infty} 1_{(x,x+\theta]}(y)dF(y) \right\} dx \\
&= \int_{-\infty}^{\infty} \left\{ \int 1_{(x,x+\theta]}(y)dx \right\} dF(y) \\
&= \int_{-\infty}^{\infty} \left\{ \int 1_{[y-\theta,y)}(x)dx \right\} dF(y) \\
&= \int_{-\infty}^{\infty} \{\theta\}\, dF(y) = \theta \cdot 1 = \theta.
\end{aligned}$$

4. PfS Exercise 4.11, page 114:
(a) Show that $\int_0^\infty \{P(|X| > x)\}^{1/2}dx < \infty$ implies $E(X^2) < \infty$.
(b) Show that $\int_0^\infty \{P(|X| > x)\}^{1/2}dx \leq \frac{r}{r-2}\|X\|_r$ for any $r > 2$ so that the integral on the left is finite whenever $X \in \mathcal{L}_r$ for any $r > 2$. If $\int_0^\infty \{P(|X| > x)\}^{1/2}dx < \infty$ then we say that $X \in \mathcal{L}_{2,1}$; this condition arises in connection with optimal transportation inequalities for empirical processes and in multiplier and bootstrap CLTs.

Solution: (a) Now $t^2 P(|X| \geq t) \leq E\{|X|^2 1_{[|X|\geq t]}\} \leq E|X|^2$ for all $t \geq 0$. Thus, using problem 1 and letting $t_0 > 0$,

$$\begin{aligned}
E|X|^2 &= \int_0^\infty 2tP(|X| > t)dt = \int_0^{t_0} 2tP(|X| > t)dt + \int_{t_0}^\infty 2tP(|X| > t)dt \\
&\leq \int_0^{t_0} 2tdt + \int_{t_0}^\infty 2\sqrt{t^2 P(|X| > t)}\sqrt{P(|X| > t)}dt \\
&\leq t_0^2 + 2\|X\|_2 \int_{t_0}^\infty \sqrt{P(|X| > t)}dt \\
&\leq 2^{-1}\|X\|_2^2 + 2\|X\|_2 \int_0^\infty \sqrt{P(|X| > t)}dt \quad \text{by choosing } t_0 = 2^{-1/2}\|X\|_2.
\end{aligned}$$

Rearranging terms yields

$$\frac{1}{2}\|X\|_2^2 \le 2\|X\|_2 \int_0^\infty \sqrt{P(|X|>t)}dt,$$

and hence

$$\|X\|_2 \le 4\int_0^\infty \sqrt{P(|X|>t)}dt.$$

(b) We proceed somewhat similarly to the proof in (a): for any fixed $t_0 \in (0,\infty)$ we have, by Markov's inequality,

$$\begin{aligned}\int_0^\infty \sqrt{P(|X|>t)}dt &= \int_0^{t_0} \sqrt{P(|X|>t)}dt + \int_{t_0}^\infty \sqrt{\frac{\|X\|_r^r}{t^r}}dt \\ &\le t_0 + \|X\|_r^{r/2}\frac{t_0^{1-r/2}}{r/2-1}.\end{aligned}$$

The right side in the last line of the last display is minimized by $t_0 \equiv \|X\|_r$, and the minimum value is

$$\|X\|_r + \|X\|_r^{r/2}\frac{\|X\|_r^{1-r/2}}{r/2-1} = \frac{r}{r-2}\|X\|_r.$$

5. PfS, Exercise 5.1.4, page 91.
Let $\mathcal{X} = [0,1]$, $\mathcal{Y} = (1,\infty)$ both equipped with the Borel sets and Lebesgue measure. Let $f(x,y) = e^{-xy} - 2e^{-2xy}$. Show that
(i) $\int_0^1 (\int_1^\infty f(x,y)dy)dx = \int_0^1 x^{-1}(e^{-x} - e^{-2x})dx$ exists and is > 0.
(ii) $\int_1^\infty (\int_0^1 f(x,y)dx)dy = \int_1^\infty y^{-1}(e^{-2y} - e^{-y})dy$ exists and is < 0.

Solution: First,

$$\int_0^1 \left(\int_1^\infty f(x,y)dy\right)dx = \int_0^1 x^{-1}\left(e^{-x} - e^{-2x}\right)dx$$

by an easy calculation of the inner integral. Note that the function $x^{-1}(e^{-x} - e^{-2x})$ converges to 1 as $x \downarrow 0$, and is continuous elsewhere, hence is uniformly continuous and uniformly bounded on $[0,1]$. Since it is strictly positive and bounded, the last integral exists and is positive. On the other hand,

$$\int_1^\infty \left(\int_0^1 f(x,y)dx\right)dy = \int_1^\infty y^{-1}(e^{-2y} - e^{-y})dy$$

again by an easy calculation of the inner integral. Now the integrand is negative (since $e^{-2y} < e^{-y}$ for all $y > 0$), bounded and the two integrals $\int_1^\infty y^{-1}e^{-2y}dy$ and $\int_1^\infty y^{-1}e^{-y}dy$ both are clearly finite. Thus the second iterated integral exists and is strictly negative. Letting $Ei(z) \equiv -\int_z^\infty t^{-1}e^{-t}dt$, the exponential integral function, it is easily shown that

$$\begin{aligned}\int_0^1 \left(\int_1^\infty f(x,y)dy\right) dx &= \int_0^1 x^{-1}\left(e^{-x} - e^{-2x}\right) dx \\ &= -Ei(-2) + Ei(-1) + \log(2) \\ &\doteq 0.522664...,\end{aligned}$$

while

$$\begin{aligned}\int_1^\infty \left(\int_0^1 f(x,y)dx\right) dy &= \int_1^\infty y^{-1}(e^{-2y} - e^{-y})dy \\ &= -Ei(-2) + Ei(-1) \\ &\doteq -0.170483....\end{aligned}$$

Of course the difficulty here is that

$$\int_1^\infty \left(\int_0^1 |f(x,y)|dx\right) dy = \int_0^1 \left(\int_1^\infty |f(x,y)|dy\right) dx = \infty\,.$$

In fact

$$\begin{aligned}\int_0^1 \left(\int_1^\infty f^+(x,y)dx\right) dy &= \int_0^1 \left(\int_{x^{-1}\log 2}^\infty f(x,y)dy\right) dx \\ &= \int_0^1 x^{-1}(e^{-\log 2} - e^{-2\log 2})dx \\ &= \frac{1}{4}\int_0^1 x^{-1}dx = +\infty,\end{aligned}$$

and similarly

$$\begin{aligned}\int_0^1 \left(\int_1^\infty f^-(x,y)dx\right) dy &= \int_0^1 \left(\int_0^{x^{-1}\log 2} f(x,y)dy\right) dx \\ &= \int_0^1 x^{-1}((1 - e^{-2\log 2}) - (1 - e^{-\log 2}))dx \\ &= \frac{1}{4}\int_0^1 x^{-1}dx = +\infty.\end{aligned}$$

See Figures 1-3 in the “long” version.