Statistics 521, Problem Set 7 Solutions

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1. Let μ be a sigma-finite measure and let ν be a finite measure on $(\Omega, \mathcal{A})$. Set $\phi = \mu - \nu$; i.e. define $\phi : \mathcal{A} \to (-\infty, \infty]$ by $\phi(A) = \mu(A) - \nu(A)$.
(a) Show that ϕ is a signed measure.
(b) Show that

$$\phi(A) = \int_A (f - g) d(\mu + \nu)$$

for some measurable functions f and g, $g \in \mathcal{L}_1(\mu + \nu)$. Thus ϕ can be written in the canonical form of the signed measure discussed in example 4.1.1 and Problem Set 6, problem 5.
(c) Apply the results of Problem Set 6, problem 9 (i.e. PfS Exercise 4.1.2, page 67) to ϕ: compute ϕ^+, ϕ^-, $|\phi|$, and $|\phi|(\Omega)$, assuming for the latter that μ is also a finite measure.

Solution: (a) First $\phi(\emptyset) = \mu(\emptyset) - \nu(\emptyset) = 0 - 0 = 0\,$,; next, for $A \in \mathcal{A}$ we have $\phi(A) = \mu(A) - \nu(A) \in (-\infty, \infty]$ since $\nu(\Omega) < \infty$ and $\mu(A), \nu(A) \geq 0$; finally, for any collection of disjoint sets $A_n \in \mathcal{A}$,

$$\begin{aligned} \phi(\sum A_n) &= \mu(\sum A_n) - \nu(\sum A_n) = \sum \mu(A_n) - \sum \nu(A_n) \\ &= \sum(\mu(A_n) - \nu(A_n)) = \sum \phi(A_n)\,. \end{aligned}$$

Thus ϕ is a signed measure.
(b) Now $\mu \ll \mu+\nu$ and $\nu \ll \mu+\nu$, so by the Radon-Nikodym theorem, $f = \frac{d\mu}{d(\mu+\nu)}$ and $g = \frac{d\nu}{d(\mu+\nu)}$ exist, are measurable, and, for $A \in \mathcal{A}$,

$$\mu(A) = \int_A f d(\mu + \nu), \qquad \nu(A) = \int_A g d(\mu + \nu)\,.$$

Hence we have

$$\phi(A) = \mu(A) - \nu(A) = \int_A (f - g) d(\mu + \nu)\,.$$

(c) Note that in part (b) we have written ϕ in the form of the signed measure treated in problem #1 with X replaced by $(f-g)$ and μ replaced by $\mu+\nu$. Hence it follows that

$$\phi^+(A) = \int_A (f-g)^+ d(\mu+\nu)\,, \qquad \phi^-(A) = \int_A (f-g)^- d(\mu+\nu)$$

and

$$|\phi|(A) = \int_A |f-g| d(\mu+\nu)\,.$$

Thus we have $|\phi|(\Omega) = \int_\Omega |f-g| d(\mu+\nu)$.

2. For probability measures P and Q on $(\Omega, \mathcal{A})$, define

$$d_{TV}(P,Q) = \sup_{A\in\mathcal{A}} |P(A)-Q(A)|\,.$$

You showed in problem 3, problem set #6 that $d_{TV}(P,Q) = (1/2)\int |p-q| d\mu$ for any measure μ dominating both P and Q; i.e. $P << \mu$, $Q << \mu$.
(a) Show that $d_{TV}(P,Q)$ does not depend on the choice of μ.
(b) Use the results of problem 1, part (c) to show that $d_{TV}(P,Q) = (1/2)|P-Q|(\Omega)$.

Solution: (a) This is actually pretty easy given the equality $d_{TV}(P,Q) \equiv \sup_{A\in\mathcal{A}} |P(A)-Q(A)| = (1/2)\int |p_\mu - q_\mu| d\mu$ for any measure μ dominating both P and Q: if μ and ν are two such measures then we have

$$\frac{1}{2}\int |p_\mu - q_\mu| d\mu = \sup_{A\in\mathcal{A}} |P(A)-Q(A)| = \frac{1}{2}\int |p_\nu - q_\nu| d\nu\,.$$

On the other hand, another proof which does not use this identity goes as follows. Since $\mu \ll \mu+\nu$ and $\nu \ll \mu+\nu$, we know that the Radon-Nikodym derivatives $f = d\mu/d(\mu+\nu)$ and $g = d\nu/d(\mu+\nu)$ exist. Furthermore, $P \ll \mu+\nu$ and $Q \ll \mu+\nu$. Hence

$$\begin{aligned} p_{\mu+\nu} &= \frac{dP}{d(\mu+\nu)} = \frac{dP}{d\mu}\frac{d\mu}{d(\mu+\nu)} = p_\mu f \\ &= \frac{dP}{d(\mu+\nu)} = \frac{dP}{d\nu}\frac{d\nu}{d(\mu+\nu)} = p_\nu g\,; \end{aligned}$$

similarly, $q_{\mu+\nu} = q_\mu f = q_\nu g$. Hence it follows that

$$\begin{aligned}
\int |p_\mu - q_\mu| d\mu &= \int |p_\mu - q_\mu| f d(\mu+\nu) \\
&= \int |p_\mu f - q_\mu f| d(\mu+\nu) \\
&= \int |p_{\mu+\nu} - q_{\mu+\nu}| d(\mu+\nu) \\
&= \int |p_\nu - q_\nu| d\nu
\end{aligned}$$

where the last inequality follows from the previous chain of equalities with μ replaced by ν and f replaced by g.
(b) From problem #1, part C, for any measure μ dominating both P and Q (e.g. $\mu = P+Q$) we have $|P-Q|(\Omega) = \int |p-q| d\mu = 2d_{TV}(P,Q)$.

3. Suppose that ϕ is a sigma-finite signed measure and $X \in L_1(|\phi|)$. The integral $\int X d\phi$ is defined by

$$\int X d\phi = \int X d\phi^+ - \int X d\phi^- \, .$$

Show that $|\int X d\phi| \le \int |X| d|\phi|$.

Solution: Note that

$$\begin{aligned}
\int_\Omega X d\phi &= \int X d\phi^+ - \int X d\phi^- \\
&= \int X^+ d\phi^+ - \int X^- d\phi^+ - (\int X^+ d\phi^- - \int X^- d\phi^-)
\end{aligned}$$

where all integrals, $\int X^+ d\phi^+$, $\int X^- d\phi^+$, $\int X^+ d\phi^-$, and $\int X^- d\phi^-$ are ≥ 0. Thus it follows that

$$\begin{aligned}
\Big| \int_\Omega X d\phi \Big| &\le \int X^+ d\phi^+ + \int X^- d\phi^+ \\
&\qquad + \int X^+ d\phi^- + \int X^- d\phi^- \\
&= \int |X| d\phi^+ + \int |X| d\phi^- \\
&= \int |X| d|\phi| \, .
\end{aligned}$$

4. PfS, Exercise 4.2.3, page 73: Flip a coin. If heads results, let X be a Uniform$(0,1)$ random variable; if tails results, let X be a Poisson(λ) random variable. The resulting distribution of X on $\mathbb{R}$ is labeled ϕ.
(a) Let μ denote Lebesgue measure on $\mathbb{R}$. Find the Lebesgue decomposition of ϕ with respect to μ; that is, write $\phi = \phi_{ac} + \phi_s$.
(b) Let ν be counting measure on $\{0,1,2,\ldots\}$. Find the Lebesgue decomposition of ϕ with respect to ν.

Solution: First let $Y \sim$ Bernoulli$(1/2)$ denote the coin toss variable with $P(Y=1) = 1/2 = P(Y=0)$. Then

$$\begin{aligned} P([X \le x] \cap [Y=1]) &= (1/2)\{(x \vee 0) \wedge 1\}, \\ P([X \le x] \cap [Y=0]) &= (1/2) \sum_{k=0}^{[x]} \exp(-\lambda) \frac{\lambda^k}{k!}, \end{aligned}$$

for $x \in \mathbb{R}$, and ϕ is the measure on $\mathbb{R}$ corresponding to the distribution function given by

$$F(x) = P(X \le x) = (1/2)\{(x \vee 0) \wedge 1\} + (1/2) \sum_{k=0}^{[x]} \exp(-\lambda) \frac{\lambda^k}{k!}.$$

(a) If μ is Lebesgue measure on $\mathbb{R}$, then the Poisson part of ϕ is singular with respect to μ, and the Uniform$(0,1)$ part is absolutely continuous with respect to μ. Thus with

$$\begin{aligned} \phi_{ac}(A) &= (1/2)\mu(A \cap [0,1]) = \int_A \frac{1}{2} 1_{(0,1)}(z) d\mu(z) = \int_A \frac{1}{2} 1_{(0,1)}(z) dz, \\ \phi_s(A) &\equiv \frac{1}{2} \sum_{k \in A} \exp(-\lambda) \frac{\lambda^k}{k!}, \end{aligned}$$

we have $\phi(A) = \phi_{ac}(A) + \phi_s(A)$ for all Borel sets A. Note that with $D \equiv \{0,1,2,\ldots\}$ we have $\phi_{ac}(D) = 0$ and $\phi_s(D^c) = 0$.
(b) If μ is counting measure on $D = \{0,1,2,\ldots\}$, then the Uniform$(0,1)$ part of ϕ is singular with respect to μ, and the Poisson part of ϕ is absolutely continuous with respect to μ. Thus with

$$\begin{aligned} \phi_s(A) &= (1/2)\mu(A \cap [0,1]) = \int_A \frac{1}{2} 1_{(0,1)}(z) dz, \\ \phi_{ac}(A) &\equiv \frac{1}{2} \sum_{k \in A} \exp(-\lambda) \frac{\lambda^k}{k!} = \int_A \frac{1}{2} e^{-\lambda} \frac{\lambda^k}{k!} d\mu(k), \end{aligned}$$

we can write $\phi(A) = \phi_{ac}(A) + \phi_s(A)$. Note that now $\phi_{ac}(D^c) = 0$ while $\phi_s(D) = 0$.

5. PfS, Exercise 4.4.3, page 84: Let F be $\nearrow$, right-continuous and bounded on $\mathbb{R}$ with $F(-\infty) = 0$. Define μ_F via $\mu_F((a, b]) = F(b) - F(a)$ for all $a < b$. Show that $\mu_F << \lambda$ if and only if F is an absolutely continuous function on $\mathbb{R}$ (in the sense of PfS, Definition 4.4.2, page 80).

   Solution: First suppose $\mu_F << \lambda$. Then by Theorem 4.2.1, page 72 (the Radon-Nikodym theorem), for all Borel sets A we have
   $$\mu_F(A) = \int_A Z_0 d\lambda.$$
   Since F is bounded and $F(-\infty) = 0$
   $$\mu_F(R) = \int_R Z_0 d\lambda < \infty, \qquad \text{and} \qquad Z_0 \geq 0.$$
   Thus $Z_0 \in \mathcal{L}_1(\lambda)$. By Theorem 3.2.5 (absolute continuity of the integral), for every $\epsilon > 0$ there exists a $\delta_\epsilon > 0$ such that $\lambda(A) < \delta_\epsilon$ implies $\mu_F(A) = \int_A Z_0 d\lambda < \epsilon$. Take $A = \sum_{k=1}^n (c_k, d_k]$ with $\lambda(A) = \sum_{k=1}^n (d_k - c_k) < \delta_\epsilon$. Then
   $$\epsilon > \int_A Z_0 d\lambda = \mu_F(A) = \mu_F(\sum_{k=1}^n (c_k, d_k]) = \sum_{k=1}^n |F(d_k) - F(c_k)|;$$
   i.e. F is an absolutely continuous function on $\mathbb{R}$.

   Conversely, suppose F is an absolutely continuous function on $\mathbb{R}$ with $F(-\infty) = 0$, $F \nearrow$, and $F(\infty) < \infty$. Then, by Theorem 4.4.1 (the fundamental theorem of calculus), F' exists a.e. λ and
   $$F(x) = F(x) - F(-\infty) = \int_{-\infty}^x F' d\lambda \qquad \text{for all } \ x \in \mathbb{R}.$$
   Thus if μ_F is the corresponding measure with $\mu_F((a, b]) = F(b) - F(a)$, it follows that for any Borel set A
   $$\mu_F(A) = \int_A F' d\lambda.$$
   But then $\mu_F << \lambda$ by the Radon-Nikodym theorem 4.2.1.

6. **Bonus problem:**
(a) Let the Hellinger distance $H(P,Q)$ be defined for probability measures P,Q on a measurable space $(\Omega,\mathcal{A})$ by

$$H^2(P,Q) \equiv \frac{1}{2}\int \{\sqrt{p}-\sqrt{q}\}^2 d\mu$$

where $p = dP/d\mu$ and $q = dQ/d\mu$ and μ is any measure dominating both P and Q: $P << \mu$ and $Q << \mu$. Show that $H(P,Q)$ does not depend on the choice of μ.
(b) Show that

$$H^2(P,Q) \le d_{TV}(P,Q) \le cH(P,Q)$$

for some absolute constant c. Find c explicitly.