Statistics 521: Problem Set 6 Solutions

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1. (a) Give an example of a sequence of random variables X_n, X (all defined on a common probability space $(\Omega, \mathcal{A}, P)$) satisfying $X_n \rightarrow_{a.s.} X$, but $E(X_n) \nrightarrow E(X)$.
(b) Give an example of a sequence of non-negative random variables X_n, X on a common probability space satisfying $E(X_n) \rightarrow E(X)$ but $X_n \nrightarrow_{a.s.} X$.
(c) Give an example of a sequence of random variables X_n, X satisfying $X_n \rightarrow_d X$, but $X_n \nrightarrow_{p,a.s.,1} X$.

Solution: (a) Let $U \sim \text{Uniform}[0,1]$. For $\alpha \geq 0$, let $X_n \equiv n^\alpha 1_{(1/(n+1),1/n]}(U)$. Then $X_n \rightarrow_{a.s.} 0 \equiv X$ since $X_n = 0$ for $n > 1/U$ and $P(U \in (0,1]) = 1$. But

$$E(X_n) = n^\alpha (n^{-1} - (n+1)^{-1}) = n^{\alpha-1}/(n+1) \rightarrow 1_{\{2\}}(\alpha) + \infty \cdot 1_{(2,\infty)}(\alpha).$$

Thus $E(X_n) \nrightarrow E(X)$ for $\alpha \geq 2$.
(b) Let $U \sim \text{Uniform}[0,1]$. For $0 < \alpha < 1$, $m \geq 1$, and $1 \leq k \leq 2^m$ define

$$Y_{m,k} \equiv (2^{m\alpha}) 1_{((k-1)/2^m, k/2^m]}(U);$$

Then let $X_n \equiv X_{2^m+k} \equiv Y_{m,k}$ for $m \geq 1$ and $1 \leq k \leq 2^m$. Note that $E(X_n) = E(Y_{m,k}) = 2^{m\alpha} \cdot 2^{-m} = 2^{(\alpha-1)m} \rightarrow 0$ as $n = 2^m + k \rightarrow \infty$, but $X_n = Y_{m,k} > 0$ i.o. with probability 1 since $P(U \in (0,1]) = 1$.
(c) Suppose that $X_1, X_2, \ldots$ are independent and identically distributed random variables with common distribution function F all defined on a common probability space. (We will make this completely rigorous in chapter 5.) Then $F_n(x) = P(X_n \leq x) = F(x)$, so F_n certainly converges to F for all $x \in \mathbb{R}$ and in particular at all $x \in C_F$. Thus $X_n \rightarrow_d X$, but $X_n \nrightarrow_{a.s.,p,1}$. Alternatively, the X_n's could be taken to be defined on separate probability spaces $(\Omega_n, \mathcal{A}_n, P_n)$ with induced distributions P_{X_n} on $\mathbb{R}$ with distribution functions $F_n(x) \equiv P(X_n \leq x)$ satisfying $F_n \rightarrow_d F$. Now we cannot even talk about the random variables $X_n - X$, so $X_n \nrightarrow_{a.s.,p,1} X$. Here is yet a further simple example: Suppose that $X_n \equiv U \sim \text{Uniform}[0,1]$ for all $n \geq 1$. Suppose

that $X \equiv 1 - U$. Then $X \sim$ Uniform$[0,1]$ so $X_n \stackrel{d}{=} X$ for all n and hence $X_n \to_d X$, while $X_n \equiv U \neq_{a.s.} 1 - U \equiv_X$, so $X_n \nrightarrow_{p,a.s.} X$. Note that

$$\begin{aligned}\{|X_n - X| \geq \epsilon\} &= \{|U - (1-U)| \geq \epsilon\} = \{|2U - 1| \geq \epsilon\} \\ &= \{(1-\epsilon)/2 \leq U \leq (1+\epsilon)/2\}^c\end{aligned}$$

has $P(|X_n - X| \geq \epsilon) = 1 - \epsilon$ for every $n \geq 1$ and hence $X_n \nrightarrow_p X$. This also implies that $X_n \nrightarrow_{a.s.} X$.

2. PfS, Exercise 3.5.7, page 61, modified as follows: Suppose that $f_0, f_1, \ldots$ are ≥ 0, defined on a sigma-finite measure space $(\Omega, \mathcal{A}, \mu)$. (a) Suppose that $\int_\Omega f_n d\mu = 1$ for $n = 0, 1, \ldots$, and $f_n \to_{a.e.} f_0$ with respect to μ. Show that

$$\sup_{A \in \mathcal{A}} \Big| \int_A f_n d\mu - \int_A f_0 d\mu \Big| \to 0 \qquad \text{as } \ n \to \infty\,.$$

(b) Show that the conclusion of (a) holds if just $f_n \to_\mu f_0$ and $\int_\Omega f_n d\mu \to \int_\Omega f_0 d\mu$.

Solution: (a) By the solution to problem #3 below,

$$\sup_{A \in \mathcal{A}} \Big| \int_A f_n d\mu - \int_A f_0 d\mu \Big| = \int (f_0 - f_n)^+ d\mu$$

where $(f_0 - f_n)^+ \to_{a.e.} 0$ and is dominated by the integrable function f_0. Hence the right side converges to 0 by the dominated convergence theorem.

(b) If we have $f_n \to_\mu f_0$ and $\int f_n d\mu \to \int f_0 d\mu$, then we still have

$$\begin{aligned} & \sup_{A\in\mathcal{A}} |\int_A f_n d\mu - \int_A f_0 d\mu| \qquad (1) \\ \leq & \sup_A \int_A |f_n - f_0| d\mu \\ \leq & \int_\Omega |f_n - f_0| d\mu = \int_\Omega |f_0 - f_n| d\mu \\ = & \int (f_0 - f_n)^+ d\mu + \int (f_0 - f_n)^- d\mu \\ = & \int (f_0 - f_n)^+ d\mu + \int (f_0 - f_n)^+ d\mu - D_n \\ = & 2\int (f_0 - f_n)^+ d\mu - D_n \qquad (2) \end{aligned}$$

where

$$D_n \equiv \int_\Omega (f_0 - f_n) d\mu = \int_\Omega \{(f_0 - f_n)^+ - (f_0 - f_n)^-\} d\mu \to 0\,.$$

But the right side of (2) converges to 0 by the dominated convergence theorem together with $D_n \to 0$.

3. Suppose that P, Q are two probability measures on the same measurable space $(\Omega, \mathcal{A})$ which are both absolutely continuous with respect to the measure μ with densities (Radon-Nikodym derivatives) p and q respectively. Thus $P(A) = \int_A p d\mu$ and $Q(A) = \int_A q d\mu$ for $A \in \mathcal{A}$. Show that

$$d_{TV}(P, Q) \equiv \sup_{A\in\mathcal{A}} |P(A) - Q(A)| = \frac{1}{2}\int |p - q| d\mu = \int (p - q)^+ d\mu\,.$$

Solution: Let $\delta = p - q$, so that $\int_\Omega \delta d\mu = 0$. Then for $A \in \mathcal{A}$ we have $0 = \int_\Omega \delta d\mu = \int_A \delta d\mu + \int_{A^c} \delta d\mu$ and hence $|\int_{A^c} \delta d\mu| = |\int_A \delta d\mu|$. Thus for $A \in \mathcal{A}$ we have

$$2|\int_A \delta d\mu| = |\int_A \delta d\mu| + |\int_{A^c} \delta d\mu| \leq \int_\Omega |\delta| d\mu\,.$$

If $A = [\delta \geq 0]$, then we have equality in the above inequality, and hence it follows that

$$\sup_{A\in\mathcal{A}} |P(A)-Q(A)| = \sup_{A\in\mathcal{A}} |\int_A (p-q)d\mu| = \frac{1}{2}\int_\Omega |p-q|d\mu = \int (p-q)^+ d\mu\,.$$

Note that the hypothesis of this problem, namely $P << \mu$ and $Q << \mu$ for some measure μ is *always satisfied* with $\mu \equiv P + Q$.

4. Suppose that $X_n \sim \text{Binomial}(n, p_n)$ for $n = 1, 2, \ldots$ with $np_n \to \lambda > 0$, and let P_n be the induced distribution of X_n on $\mathbb{R}$. Let $X_0 \sim \text{Poisson}(\lambda)$ and let P_0 be the corresponding induced distribution on $\mathbb{R}$. Use Scheffé's theorem to show that $d_{TV}(P_n, P_0) \to 0$ as $n \to \infty$.

   Solution: As we showed in problem 5, Problem Set 1,

$$p_n(k) \equiv P(X_n = k) = \binom{n}{k} p_n^k (1-p_n)^{n-k} \to \frac{\lambda^k}{k!}e^{-\lambda} \equiv p_0(k)$$

   for each fixed $k \geq 0$. Thus the hypotheses of problem 2(a) hold and we conclude that

$$d_{TV}(P_n, P_0) \equiv \sup_{A\in 2^{\mathbb{N}}} |P_n(A) - P_0(A)| \to 0.$$

   This is considerably stronger than $X_n \to_d X_0 \sim \text{Poisson}(\lambda)$.

5. Let $X_{n1}, \ldots, X_{nn}$ be independent, $X_{nk} \sim \text{Bernoulli}(p_{nk})$, and let $Y_n \sim \text{Poisson}(\sum_{k=1}^n p_{nk})$. Let P_n be the distribution of $\sum_{k=1}^n X_{nk}$ and let Q_n be the distribution of Y_n. Show that

$$d_{TV}(P_n, Q_n) \equiv \sup_{A\in\mathcal{B}} |P(S_n \in A) - P(Y_n \in A)| \leq \sum_{k=1}^n p_{nk}^2.$$

   Note that when $p_{nk} = p_n \to 0$ for all k and $np_n \to \lambda$, then $\sum_{k=1}^n p_{nk}^2 = np_n^2 = (np_n)^2/n = O(n^{-1})$.

   Hint: Construct S_n and Y_n on a common probability space as follows: let $T_{nk} \sim \text{Poisson}(p_{nk})$, $k = 1, \ldots, n$ be independent, and let $Z_{nk} \sim \text{Bernoulli}(1 - (1 - p_{nk})e^{p_{nk}})$, $k = 1, \ldots, n$ be independent and independent of the T_{nk}'s. Define $X_{nk} = 1_{[T_{nk}\geq 1]} + 1_{[T_{nk}=0]}1_{[Z_{nk}=1]}$. Set

$S_n = \sum_{k=1}^n X_{nk}$, $Y_n = \sum_{k=1}^n T_{nk}$. Check that $X_{nk} \sim \text{Bernoulli}(p_{nk})$, $Y_n \sim \text{Poisson}(\sum_1^n p_{nk})$, and

$$\begin{aligned} P(T_{nk} = 0, X_{nk} = 1) &= e^{-p_{nk}} - (1 - p_{nk}) \\ P(T_{nk} \geq 1, X_{nk} = 0) &= 0, \quad P(T_{nk} \geq 2) = 1 - e^{-p_{nk}} - p_{nk} e^{-p_{nk}}. \end{aligned}$$

Show that

$$d_{TV}(P_n, Q_n) \leq P(S_n \neq Y_n) \leq \sum_{k=1}^n P(X_{nk} \neq T_{nk}) \leq \sum_{k=1}^n p_{nk}^2.$$

Solution: Using the notation in the hint we first show that $X_{nk} \sim \text{Bern}(p_{nk})$: this follows since, using $P(Y_\lambda \geq 1) = 1 - P(Y_\lambda = 0) = 1 - e^{-\lambda}$ if $Y_\lambda \sim \text{Poisson}(\lambda)$,

$$\begin{aligned} P(X_{nk} = 1) &= P(T_{nk} \geq 1) + P(T_{nk} = 0) P(Z_{nk} = 1) \\ &= 1 - e^{-p_{nk}} + e^{-p_{nk}} (1 - (1 - p_{nk}) e^{p_{nk}}) \\ &= p_{nk}. \end{aligned}$$

Next,

$$P(T_{nk} = 0, X_{nk} = 1) = e^{-p_{nk}} (1 - (1 - p_{nk}) e^{p_{nk}} = e^{-p_{nk}} - (1 - p_{nk}),$$

while

$$P(T_{nk} \geq 1, X_{nk} = 0) = P(T_{nk} \geq 1, \ T_{nk} = 0) = 0,$$

and

$$P(T_{nk} \geq 2) = 1 - P(T_{nk} = 0 \text{ or } 1) = 1 - e^{-p_{nk}} - p_{nk} e^{-p_{nk}}.$$

Thus

$$\begin{aligned} P(X_{nk} \neq T_{nk}) &= P(X_{nk} = 0, T_{nk} = 1) + P(X_{nk} = 1, T_{nk} = 0) + P(T_{nk} \geq 2) \\ &= 0 + e^{-p_{nk}} - (1 - p_{nk}) + 1 - e^{-p_{nk}} - p_{nk} e^{-p_{nk}} \\ &= p_{nk} (1 - e^{-p_{nk}}) \leq p_{nk}^2. \end{aligned}$$

Thus for any $A \in 2^{\mathbb{N}}$,

$$\begin{aligned}
& P(S_n \in A) - P(Y_n \in A) \\
&= P([S_n \in A] \cap [S_s = Y_n]) + P([S_n \in A] \cap [S_n \neq Y_n]) \\
&\quad - P([Y_n \in A] \cap [S_n = Y_n]) + P([Y_n \in A] \cap [S_n \neq Y_n]) \\
&= P([S_n \in A] \cap [S_n \neq Y_n]) - P([Y_n \in A] \cap [S_n \neq Y_n]) \\
&\leq P([S_n \in A] \cap [S_n \neq Y_n]) \leq P(S_n \neq Y_n).
\end{aligned}$$

Similarly, by a symmetric argument,

$$P(S_n \in A) - P(Y_n \in A) \geq -P(S_n \neq Y_n),$$

and this yields

$$\begin{aligned}
d_{TV}(P_n, Q_n) &\equiv \sup_{A \in 2^{\Omega}} |P(S_n \in A) - P(Y_n \in A))| \\
&\leq P(S_n \neq Y_n) \leq \sum_{k=1}^{n} P(X_{nk} \neq T_{nk}) \leq \sum_{k=1}^{n} p_{nk}^2.
\end{aligned}$$

If $p_{nk} = \lambda/n$ for $1 \leq k \leq n$ for some $\lambda > 0$, then this bound yields

$$d_{TV}(P_n, Q_n) \leq n(\lambda/n)^2 = \frac{\lambda^2}{n}.$$

For still stronger results, see Barbour, Holst, and Janson (1992), *Poisson Approximation*.

6. Let $X_1, X_2, \ldots, X_n, \ldots$ be i.i.d. Uniform$(0,1)$ random variables. Let $Y_n \equiv nX_{n:1} \equiv n \min_{1 \leq i \leq n} X_i$ be the first order statistic of the first n of the X_i's.
(a) Compute the survival function $1 - F_{Y_n}$ of Y_n and show that $Y_n \to_d Y \sim$ exponential(1).
(b) Compute the density function f_{Y_n} of Y_n and show that $f_{Y_n}(y) \to f_Y(y) = e^{-y}$ for $y \geq 0$.
(c) Use (b) and Scheffé's theorem to show that

$$d_{TV}(P_{Y_n}, P_Y) = \frac{1}{2} \int_0^{\infty} |f_{Y_n}(y) - f_Y(y)| dy \to 0.$$

How fast is the convergence in the last display?

Solution: (a) For $y \geq 0$ we have

$$\begin{aligned} 1 - F_{Y_n}(y) &= P(Y_n > y) = P(X_{n:1} > y/n) \\ &= P(X_1 > y/n, \ldots, Y_n > y/n) = P(X_1 > y/n) \cdots P(X_n > y/n) \\ &= P(X_1 > y/n)^n = (1 - y/n)_+^n \end{aligned}$$

where $z_+ \equiv z1\{z \geq 0\}$.
(b) It follows from the last line of (a) that

$$1 - F_{Y_n}(y) = (1 - y/n)_+^n \to e^{-y} = P(Y > y) \;\; \text{for all } y \geq 0$$

where $Y \sim$ Exponential(1). Thus $Y_n \to_d Y$.
(b) The density function of Y_n is just

$$\begin{aligned} f_{Y_n}(y) &= F'_{Y_n}(y) = (1 - y/n)_+^{n-1} 1_{[0,\infty)}(y) \\ &\to e^{-y} = f_Y(y) \quad \text{for all } \;\; y > 0. \end{aligned}$$

(c) The hypotheses of Scheffé's theorem are satisfied by (b), so we conclude that

$$d_{TV}(P_{Y_n}, P_Y) = \frac{1}{2} \int_0^\infty \left| f_{Y_n}(y) - f_Y(y) \right| dy \to 0.$$

To start to get a handle on how fast the right side in the last display converges to 0, first note that

$$n(f_{Y_n}(y) - f_Y(y)) \to \frac{1}{2} y(2 - y) e^{-y} \equiv h(y)$$

for each fixed $y \in (0, \infty)$ where $\int_0^\infty |h(y)| dy = 4e^{-2}$. Thus if we can justify the interchange of limit and integration it would follow that

$$\begin{aligned} n d_{TV}(P_{Y_n}, P_Y) &= \frac{n}{2} \int_0^\infty \left| f_{Y_n}(y) - f_Y(y) \right| dy \\ &\to \frac{1}{2} \int_0^\infty \frac{1}{2} |y(2 - y)| e^{-y} dy \\ &= 2e^{-2}. \end{aligned}$$

I suspect that this limiting relation can be turned into an exact bound with a little more work.

7. Suppose that $g : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ is convex. Show that $h(x,t) \equiv tg(x/t)$ is a convex function on $\mathbb{R}^d \times (0,\infty)$.
Hint: First show that $h(cx, st) = ch(x,t)$ for any $c > 0$, $(x,t) \in \mathbb{R}^d \times (0,\infty)$, and hence $t^{-1}h(x,t) = h(x/t, 1)$.

Solution: First note that if $c > 0$, then $h(cx, ct) = ctg(cx/ct) = ch(x,t)$, so with $c = 1/t$ it follows that $g(x/t) = h(x/t, 1) = t^{-1}h(x,t)$. Now let $\lambda \in [0,1]$ and $(x_j, t_j) \in \mathbb{R}^d \times (0,\infty)$ for $j = 1, 2$. Then

$$\begin{aligned} h(\lambda x_1 + \overline{\lambda} x_2, \lambda t_1 + \overline{\lambda} t_2) &= (\lambda t_1 + \overline{\lambda} t_2) g\left(\frac{\lambda x_1 + \overline{\lambda} x_2}{\lambda t_1 + \overline{\lambda} t_2}\right) \\ &= (\lambda t_1 + \overline{\lambda} t_2) g\left(\frac{\lambda t_1 (x_1/t_1) + \overline{\lambda} t_2 (x_2/t_2)}{\lambda t_1 + \overline{\lambda} t_2}\right) \\ &\le (\lambda t_1 + \overline{\lambda} t_2)\left\{\frac{\lambda t_1}{\lambda t_1 + \overline{\lambda} t_2} g(x_1/t_1) + \frac{\overline{\lambda} t_2}{\lambda t_1 + \overline{\lambda} t_2} g(x_2/t_2)\right\} \\ &= \lambda t_1 g(x_1/t_1) + \overline{\lambda} t_2 g(x_2/t_2) = \lambda h(x_1, t_1) + \overline{\lambda} h(x_2, t_2). \end{aligned}$$

It follows that h is convex.

8. For $s \in \mathbb{R} \cup \{\pm\infty\}$, $u, v \in \mathbb{R}^+$, and $\lambda \in [0,1]$, the Hölder mean (or generalized mean) $M_s(u, v; \lambda)$ of order s is defined by

$$M_s(u,v;\lambda) = \begin{cases} (\lambda u^s + (1-\lambda) v^s)^{1/s}, & s \neq 0,\ u, v > 0, \\ 0, & s < 0,\ uv = 0, \\ u^\lambda v^{1-\lambda}, & s = 0, \\ u \wedge v, & s = -\infty, \\ u \vee v, & s = +\infty. \end{cases}$$

(a) Interpret $M_s(u, v; \lambda)$ in terms of some function of the expected value of some random variable X.
(b) Show that for any $r < s$ the inequality $M_r(u,v;\lambda) \le M_s(u,v;\lambda)$ holds for all $u, v \in \mathbb{R}$, $\lambda \in [0,1]$. Thus

$$M_r(u,v;\lambda) \le M_0(u,v;\lambda) \le M_s(u,v;\lambda).$$

(In class on 10/31 we proved a related statement with $r = -1$ and $s = 1$.)

Solution: (a) Let X be a random variable taking on the value u with probability λ and v with probability $1-\lambda$. Thus for $u, v > 0$ and $s \neq 0$ we have

$$M_s(u, v; \lambda) = \{E(X^s)\}^{1/s} \equiv \|X\|_s. \qquad (3)$$

For $s = 0$ and $u, v > 0$

$$M_0(u, v; \lambda) = \exp\left(E \log X\right).$$

(b) For $0 < r < s$ the inequality $M_r(u, v; \lambda) \leq M_s(u, v; \lambda)$ becomes $\|X\|_r \leq \|X\|_s$ in view of (3), and this is just Liapunov's inequality Inequality 3.4.4, PfS page 48.) For $r < s < 0$, the inequality follows by replacing X by $1/X$ and $r < s < 0$ by $0 < -s < -r$. Now consider $r = 0$ and $0 < s$. By concavity of $g(y) = s^{-1} \log y$ it follows from Jensen's inequality that

$$E \log X = E s^{-1} \log X^s \leq s^{-1} \log E(X^s) = \log(\|X\|_s),$$

and hence

$$M_0(u, v; \lambda) = E \exp(E \log X) \leq \|X\|_s = M_s(u, v; \lambda)$$

For the case $r < 0 = s$, note that $g(y) = r^{-1} \log y$ is convex, so Jensen's inequality gives

$$E \log X = E(r^{-1} \log X^r) \geq r^{-1} \log X^r = \log(\|X\|_r),$$

and hence

$$M_0(u, v; \lambda) = E \exp(E \log X) \geq \|X\|_r = M_r(u, v; \lambda).$$

The cases with $r = -\infty$ or $s = \infty$ are easy. When $r < \infty$ and $s = +\infty$,

$$\begin{aligned} \{EX^r\}^{1/r} &\leq \{EX^t\}^{1/t} = \{\lambda u^t + \overline{\lambda} v^t\}^{1/t} \quad \text{for every} \quad r < t < \infty \\ &\leq \{\lambda (u \vee v)^t + \overline{\lambda} (u \vee v)^t\}^{1/t} = u \vee v = M_\infty(u, v; \lambda). \end{aligned}$$

The case $r = -\infty$ and $s > -\infty$ is similar.

9. PfS, Exercise 4.1.2, page 67: Identify ϕ^+, ϕ^-, $|\phi|$ and $|\phi|(\Omega)$ in the context of the prototypical situation of example 4.1.1, page 66. Be sure to specify Ω^+ and Ω^-.

Solution: I claim that

$$\begin{aligned}
\phi^+(A) &= \int_A X^+ d\mu = \phi(A\Omega^+) \;\text{ with }\; \Omega^+ = \{\omega : X(\omega) \geq 0\}, \\
\phi^-(A) &= \int_A X^- d\mu = -\phi(A\Omega^-) \;\text{ with }\; \Omega^- = \{\omega : X(\omega) < 0\}, \\
|\phi|(A) &= \int_A |X| d\mu, \qquad \text{and} \\
|\phi|(\Omega) &= \int |X| d\mu.
\end{aligned}$$

To see this, note that Ω^+, Ω^- are, respectively, positivity, negativity sets for ϕ since

$$\begin{aligned}
\phi(A) &= \int_A X d\mu \geq 0 \qquad \text{for all events } \; A \subset \Omega^+, \\
\phi(A) &= \int_A X d\mu \leq 0 \qquad \text{for all events } \; A \subset \Omega^-.
\end{aligned}$$

Furthermore, if $\tilde{\Omega}^+$, $\tilde{\Omega}^-$ denote the decomposition guaranteed by the Jordan-Hahn theorem 1.1, then

$$\begin{aligned}
\phi(\Omega^+ \setminus \tilde{\Omega}^+) &= \phi(\Omega^+ \cap \tilde{\Omega}^-) = 0, \qquad \text{and} \\
\phi(\tilde{\Omega}^+ \setminus \Omega^+) &= \phi(\tilde{\Omega}^+ \cap \Omega^-) = 0,
\end{aligned}$$

where the zeroes follow by using the definitions of Ω^+, Ω^-, $\tilde{\Omega}^+$, $\tilde{\Omega}^-$. Thus

$$|\phi|(\Omega^+ \Delta \tilde{\Omega}^+) = 0;$$

i.e. $\Omega^+ = [X \geq 0]$ differs from $\tilde{\Omega}^+$ by (at most) a set of $|\phi|-$measure 0.