

Statistics 521, Problem Set 4 Solutions

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1. PfS, Exercise 2.3.4, page 32: (a) Suppose that $\mu(\Omega) < \infty$ and g is continuous a.e. μ_X (that is, g is continuous except perhaps on a set of μ_X measure 0). Then $X_n \rightarrow_\mu X$ implies that $g(X_n) \rightarrow_\mu g(X)$.
(b) Let g be uniformly continuous on the real line. Then $X_n \rightarrow_\mu X$ implies that $g(X_n) \rightarrow_\mu g(X)$. (Here $\mu(\Omega) = \infty$ is allowed.)

Solution: (a) Let $\{n'\}$ be a subsequence. We want to show that for some subsequence $\{n''\}$ it follows that $g(X_{n''}) \rightarrow_{a.e.} g(X)$. Then by (15) of Theorem 2.3.1 it follows that $g(X_n) \rightarrow_\mu g(X)$. But since $X_n \rightarrow_\mu X$ we know, by (15) of Theorem 2.3.1, that there is a further subsequence $\{n''\}$ such that $X_{n''} \rightarrow_{a.e.} X$. For this subsequence we have $g(X_{n''}) \rightarrow_{a.e.} g(X)$ (by restricting in addition to the set $[X \in C_g]$ with $\mu([X \in C_g^c]) = \mu_X(C_g^c) = 0$ for which g is continuous). Thus we conclude that $g(X_n) \rightarrow_\mu g(X)$.

(b) Let $\epsilon > 0$. Since g is uniformly continuous there is a $\delta = \delta_\epsilon$ such that $|y - x| < \delta_\epsilon$ implies $|g(y) - g(x)| < \epsilon$. Since $X_n \rightarrow_\mu X$, for every $\gamma > 0$ there exists an $N = N_{\epsilon, \gamma}$ such that

$$\mu([|X_n - X| \geq \delta_\epsilon]) < \gamma, \quad \text{for all } n \geq N_{\epsilon, \gamma}.$$

Then we have

$$\begin{aligned} & \mu([|g(X_n) - g(X)| \geq \epsilon]) \\ &= \mu([|g(X_n) - g(X)| \geq \epsilon] \cap [|X_n - X| \geq \delta_\epsilon]) \\ & \quad + \mu([|g(X_n) - g(X)| \geq \epsilon] \cap [|X_n - X| < \delta_\epsilon]) \\ &\leq \mu([|X_n - X| \geq \delta_\epsilon]) + \mu(\emptyset) \\ &\leq \gamma + 0 = \gamma \quad \text{for } n \geq N_{\epsilon, \gamma}. \end{aligned}$$

Thus $\mu([|g(X_n) - g(X)| \geq \epsilon]) \rightarrow 0$ as $n \rightarrow \infty$; i.e. $g(X_n) \rightarrow_\mu g(X)$.

2. PfS, Exercise 3.2.1, page 42: Show that $X \geq 0$ and $\int X d\mu = 0$ implies $\mu([X > 0]) = 0$.

Solution: Let $\epsilon > 0$. Then $X \geq \epsilon 1_{[X \geq \epsilon]}$, and hence

$$0 = \int X d\mu \geq \epsilon \mu([X \geq \epsilon])$$

Since $[X > 0] = \cup_{n=1}^{\infty} [X \geq 1/n] = \lim [X \geq 1/n]$, We find that $\mu([X > 0]) = \lim_n \mu([X \geq 1/n]) = \lim_n 0 = 0$.

3. PfS, Exercise 3.2.2, page 42: Show that

$$\int_A X d\mu = \begin{cases} = 0, \\ \geq 0, \end{cases} \quad \text{for all } A \in \mathcal{A} \text{ implies } X = \begin{cases} = 0 \text{ a.e.}, \\ \geq 0 \text{ a.e.} \end{cases}$$

Solution: Suppose first that $\int_A X d\mu = 0$ for all $A \in \mathcal{A}$. Then with $A = [X^+ \geq 0]$ we have $0 = \int X 1_{[X^+ \geq 0]} d\mu = \int Y d\mu$ where $Y \equiv X 1_{[X^+ \geq 0]} = X^+ \geq 0$. Then by the previous exercise, $0 = \mu([Y > 0]) = \mu([X^+ > 0])$; i.e $X^+ = 0$ a.e. Similarly, choosing $A = [X^- \geq 0]$ yields $X^- = 0$ a.e.; combining the two results gives $X = X^+ - X^- = 0$ a.e.

Now suppose that $\int_A X d\mu \geq 0$ for all $A \in \mathcal{A}$. Taking $A = [X < 0] = [X^- > 0]$ yields $0 \leq \int_A X d\mu = \int -X^- d\mu \leq 0$ since $X^- \geq 0$. Thus $\int X^- d\mu = 0$. By problem 2 this implies $\mu([X < 0]) = \mu([X^- > 0]) = 0$. Hence $X \geq 0$ a.e. μ .

4. PfS, Exercise 3.2.4, page 43. Let $Y \equiv g(X)$ in the context of Theorem 3.2.6 (the ‘‘Theorem of the unconscious statistician’’). Show that the second equality holds in:

$$\int_{X^{-1}(g^{-1}(B))} g(X(\omega)) d\mu(\omega) = \int_{g^{-1}(B)} g(x) d\mu_X(x) = \int_B y d\mu_Y(y) \quad \text{for } B \in \overline{\mathcal{B}}$$

where μ_Y is the induced measure of Y on $(\overline{R}, \overline{\mathcal{B}})$.

Solution: This boils down to what is proved in the first equality with an appropriate identification of terms. Thus (Ω, \mathcal{A}) is replaced by (Ω', \mathcal{A}') , (Ω', \mathcal{A}') is replaced by $(\overline{R}, \overline{\mathcal{B}})$, g is replaced by the identity function $h : (\Omega', \mathcal{A}') \mapsto (\overline{R}, \overline{\mathcal{B}})$ given by $h(v) = v$, and X is replaced by $g : (\Omega', \mathcal{A}') \mapsto (\overline{R}, \overline{\mathcal{B}})$, and μ and μ_X are replaced by μ_X and μ_Y respectively. Thus we think of the identity

$$\int g(X(\omega)) d\mu(\omega) = \int g(x) d\mu_X(x)$$

as being replaced by

$$\int h(g(x))d\mu_X(x) = \int h(y)d\mu_{g(X)}(y) = \int h(y)d\mu_Y(y) \quad (1)$$

or, when $h(v) = v$, this yields

$$\int g(x)d\mu_X(x) = \int yd\mu_Y(y)$$

where $Y \equiv g(X)$. With this set of identifications the slightly more general identity (1) follows from the first equality, and the desired second equality is given by the special case $h(v) = v$.

5. (i) PFS, Exercise 3.3, page 45, part (a).

Suppose that the function $X(t, \cdot)$ is an integrable function on (Ω, μ) for each $t \in [a, b]$. Suppose also that for a.e. ω the partial derivative $(\partial/\partial t)X(t, \omega)$ exists for all t in the nondegenerate interval $[a, b]$ (use one-sided derivatives at the end points), and that

$$\left| \frac{\partial}{\partial t} X(t, \omega) \right| \leq Y(\omega) \quad \text{for all } t \in [a, b], \quad \text{where } Y \in \mathcal{L}_1.$$

Then the derivative and integral may be interchanged:

$$\frac{d}{dt} \int_{\Omega} X(t, \omega) d\mu(\omega) = \int_{\Omega} \frac{\partial}{\partial t} X(t, \omega) d\mu(\omega) \quad \text{for all } t \in [a, b].$$

(ii) Suppose that μ is Lebesgue measure on the unit interval $[0, 1]$ and that $(a, b) = (0, 1)$ in Exercise 3.3. If $X(t, \omega) = 1_{[\omega \leq t]}$, then for each t , $(\partial/\partial t)X(t, \omega) = 0$ almost everywhere. But $\int X(t, \omega) d\mu(\omega)$ does not differentiate to 0. Why is this not a contradiction?

Solution: (i) We write $h(t) \equiv \int_{\Omega} X(t, \omega) d\mu(\omega)$. Then by linearity of the integral and the fundamental theorem of calculus it follows that

$$\begin{aligned} h(t + \epsilon) - h(t) &= \int_{\Omega} (X(t + \epsilon, \omega) - X(t, \omega)) d\mu(\omega) \\ &= \int_{\Omega} \left(\int_t^{t+\epsilon} \frac{\partial}{\partial s} X(s, \omega) ds \right) d\mu(\omega), \end{aligned}$$

and hence that

$$\frac{h(t + \epsilon) - h(t)}{\epsilon} = \int_{\Omega} \left(\frac{1}{\epsilon} \int_t^{t+\epsilon} \frac{\partial}{\partial s} X(s, \omega) ds \right) d\mu(\omega)$$

where the integrand $\epsilon^{-1} \int_t^{t+\epsilon} \frac{\partial}{\partial s} X(s, \omega) ds$ satisfies

$$\begin{aligned} \left| \epsilon^{-1} \int_t^{t+\epsilon} \frac{\partial}{\partial s} X(s, \omega) ds \right| &\leq \frac{1}{\epsilon} \int_t^{t+\epsilon} \left| \frac{\partial}{\partial s} X(s, \omega) \right| ds \\ &\leq \frac{1}{\epsilon} \int_t^{t+\epsilon} Y(\omega) ds \text{ if } [t, t + \epsilon] \subset (a, b) \\ &= Y(\omega) \in \mathcal{L}_1, \end{aligned}$$

and where

$$\lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} \int_t^{t+\epsilon} \frac{\partial}{\partial s} X(s, \omega) ds = \lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} (X(t + \epsilon, \omega) - X(t, \omega)) = \frac{\partial}{\partial t} X(t, \omega)$$

for a.e. ω . Thus it follows from the dominated convergence theorem that

$$\lim_{\epsilon \downarrow 0} \frac{h(t + \epsilon) - h(t)}{\epsilon} = \int_{\Omega} \frac{\partial}{\partial t} X(t, \omega) d\mu(\omega).$$

By using one-sided derivatives, the same argument works at the end-points a and b of the interval $[a, b]$. Thus the claimed equality holds for all $t \in [a, b]$.

(ii) For $X(t, \omega) = 1_{[\omega \leq t]}$ and μ Lebesgue measure on $[0, 1]$ we have

$$\int_{(0,1)} X(t, \omega) d\omega = \int_{(0,1)} 1_{[\omega \leq t]} d\omega = t,$$

so

$$\begin{aligned} \frac{\partial}{\partial t} \int_{(0,1)} X(t, \omega) d\omega &= \frac{\partial}{\partial t} t = 1 \\ &\neq 0 = \int_{(0,1)} \frac{\partial}{\partial t} X(t, \omega) d\omega. \end{aligned}$$

This does not contradict (i) above because the derivative $(\partial/\partial t)1_{[\omega \leq t]}$ does not exist at $t = \omega$, and hence (i) does not apply.