Math/Stat 521, Problem Set 1 Solutions

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1. (a) Suppose that $\{\mathcal{A}_n\}$ is an increasing sequence of algebras, i.e. $\mathcal{A}_n \subset \mathcal{A}_{n+1}$ for all $n \geq 1$. Show that $\cup_{n=1}^{\infty} \mathcal{A}_n$ is an algebra.
(b) Suppose that the $\mathcal{A}_n$ of (a) are $\sigma-$algebras. Show by constructing a counter-example that $\cup_{n=1}^{\infty} \mathcal{A}_n$ need not be a $\sigma-$algebra.

   Solution: (a) If $A \in \cup_{n=1}^{\infty} \mathcal{A}_n$, then $A \in \mathcal{A}_m$ for some m, and since $\mathcal{A}_m$ is an algebra, $A^c \in \mathcal{A}_m$. Hence $A^c \in \cup_{n=1}^{\infty} \mathcal{A}_n$. If $A, B \in \cup_{n=1}^{\infty} \mathcal{A}_n$, then $A \in \mathcal{A}_m$ for some m and $B \in \mathcal{A}_n$ for some n. Without loss we can assume that $m \leq n$, and since $\mathcal{A}_m \subset \mathcal{A}_n$ it follows that $A, B \in \mathcal{A}_n$. Since $\mathcal{A}_n$ is an algebra, it follows that $A \cup B \in \mathcal{A}_n$, and hence that $A \cup B \in \cup_{n=1}^{\infty} \mathcal{A}_n$.
(b) Take $\Omega = [0,1]$. Let $\mathcal{A}_1 = \{\emptyset, \Omega\}$, $\mathcal{A}_2 = \sigma[\mathcal{A}_0, [0, 1/2)]$, $\ldots$, $\mathcal{A}_n = \sigma[\mathcal{A}_{n-1}, [0, 1 - 1/n)]$, $\ldots$. Then $\mathcal{A}_n \subset \mathcal{A}_{n+1}$ by construction, but $\cup_{n=1}^{\infty} \mathcal{A}_n$ is not a sigma field: if we let $A_k = [0, 1 - 1/k)$ for each $k = 1, 2, \ldots$, then $A_k \in \cup_{n=1}^{\infty} \mathcal{A}_n$ since $A_k \in \mathcal{A}_k$ by construction, but $[0,1) = \cup_{k=1}^{\infty} A_k \notin \cup_{n=1}^{\infty} \mathcal{A}_n$.

2. Proposition 1.1(b), PfS, page 3: There exists a minimal field, $\sigma-$field, or monotone class generated by (or containing) any specified class $\mathcal{C}$ of subsets of Ω.

   Solution: By proposition 1.1.1(i), arbitrary intersections of fields, $\sigma-$fields, or monotone classes are again fields, $\sigma-$fields, or monotone classes. Hence

$$\phi[\mathcal{C}] \equiv \bigcap \{\mathcal{A}_\alpha : \mathcal{A}_\alpha \text{ is a -field of subsets of } \Omega \text{ for which } \mathcal{C} \subset \mathcal{A}_\alpha\}$$

   is again a field, and it is the smallest such field: if $\mathcal{D}$ is the minimal field containing $\mathcal{C}$ so that $\mathcal{D} \subset \phi[\mathcal{C}]$, then we also have $\phi[\mathcal{C}] \subset \mathcal{D}$ by construction of $\phi[\mathcal{C}]$, and hence $\phi[\mathcal{C}] = \mathcal{D}$. The argument is the same for $\sigma-$fields and monotone classes with $\phi[\mathcal{C}]$ replaced by $\sigma[\mathcal{C}]$ and mon$[\mathcal{C}]$ respectively.

3. PfS, Exercise 1.1.1, PfS, page 4: Let $\mathcal{C}_1$ and $\mathcal{C}_2$ denote two collections of subsets of the set Ω. If $\mathcal{C}_1 \subset \sigma[\mathcal{C}_2]$ and $\mathcal{C}_2 \subset \sigma[\mathcal{C}_1]$, then $\sigma[\mathcal{C}_1] = \sigma[\mathcal{C}_2]$.

   Solution: Since $\mathcal{C}_1 \subset \sigma[\mathcal{C}_2]$, it follows immediately that $\sigma[\mathcal{C}_1] \subset \sigma[\sigma[\mathcal{C}_2]] = \sigma[\mathcal{C}_2]$. By a symmetric argument $\sigma[\mathcal{C}_2] \subset \sigma[\mathcal{C}_1]$. Hence $\sigma[\mathcal{C}_2] = \sigma[\mathcal{C}_1]$.

4. PfS, Exercise 1.1.2, PfS, page 8. We always have $\mu(\liminf A_n) \leq \liminf \mu(A_n)$, while $\limsup \mu(A_n) \leq \mu(\limsup A_n)$ if $\mu(\Omega) < \infty$.

   Solution: First note that $\liminf A_n = \cup_{n=1}^{\infty} \cap_{k=n}^{\infty} A_k = \cup_{n=1}^{\infty} B_n$ where $B_n = \cap_{k=n}^{\infty} A_k$ is $\uparrow$ since $B_n = \cap_{k=n}^{\infty} A_k \subset \cap_{k=n+1}^{\infty} A_k = B_{n+1}$ for all n. Hence by Proposition 1.2(i),

$$
\begin{aligned}
\mu(\liminf A_n) &= \mu(\cup_{n=1}^{\infty} B_n) \\
&= \lim_{n\to\infty} \mu(B_n) \\
&= \lim_{n\to\infty} \mu(\cap_{k=n}^{\infty} A_k) \\
&\leq \lim_{n\to\infty} \inf_{m\geq n} \mu(A_m) \\
&= \liminf \mu(A_n)
\end{aligned}
$$

   since $\cap_{k=n}^{\infty} A_k \subset A_m$ for each $m \geq n$ so that

$$
\mu(\cap_{k=n}^{\infty} A_k) \leq \mu(A_m)
$$

   for each $m \geq n$ and also $\mu(\cap_{k=n}^{\infty} A_k) \leq \inf_{m\geq n} \mu(A_m)$.
   Similarly, $\limsup A_n = \cap_{n=1}^{\infty} \cup_{k=n}^{\infty} A_k = \cap_{n=1}^{\infty} B_n$ where $B_n = \cup_{k=n}^{\infty} A_k$ is $\downarrow$ since $B_n = \cup_{k=n}^{\infty} A_k \supset \cup_{k=n+1}^{\infty} A_k = B_{n+1}$. Thus by Proposition 1.1.2(j), if $\mu(\Omega) < \infty$,

$$
\begin{aligned}
\mu(\limsup A_n) &= \mu(\cap_{n=1}^{\infty} B_n) \\
&= \lim_{n\to\infty} \mu(B_n) \\
&= \lim_{n\to\infty} \mu(\cup_{k=n}^{\infty} A_k) \\
&\geq \lim_{n\to\infty} \sup_{m\geq n} \mu(A_m) \\
&= \limsup \mu(A_n)
\end{aligned}
$$

   since $\cup_{k=n}^{\infty} A_k \supset A_m$ for each $m \geq n$ so that

$$
\mu(\cup_{k=n}^{\infty} A_k) \geq \mu(A_m)
$$

   for each $m \geq n$ and also $\mu(\cup_{k=n}^{\infty} A_k) \geq \sup_{m\geq n} \mu(A_m)$.

5. PfS(2000), Exercise 9.1.4, page 182; or PfS(2012), Exercise A.1.4, page 428. Suppose that $X_n \sim \text{Binomial}(n, p_n)$ where $np_n \to \lambda > 0$. Show that

$$P(X_n = k) \to \frac{\lambda^k}{k!}\exp(-\lambda) = P(Y = k)$$

where $Y \sim \text{Poisson}(\lambda)$; this implies that $X_n \to_d Y$. Can this be strengthened?

Solution:

$$\begin{aligned} P(X_n = k) &= \binom{n}{k} p_n^k (1 - p_n)^{n-k} \\ &= \frac{n(n-1)\cdots(n-k+1)}{k! n^k}(np_n)^k(1 - \frac{np_n}{n})^{n-k} \\ &\to \frac{1}{k!}\lambda^k e^{-\lambda} \end{aligned}$$

since $(1 + x_n/n)^n \to e^x$ if $x_n \to x$. This can be strengthened in several ways as we will see later. In particular, with $Q_n(A) = P(X_n \in A)$ and $P_\lambda(Y \in A)$ for any $A \in 2^{\mathbb{N}}$, this pointwise convergence (for each fixed k) implies that $d_{TV}(Q_n, P_\lambda) \equiv (1/2)\sum_{k=0}^{\infty} |Q_n(\{k\}) - P_\lambda(\{k\})| \to 0$; this follows from Scheffé's theorem. See Durrett pages 98 and 147 - 151 for a further generalization and strengthening via a coupling argument due to Hodges and Le Cam (1960).

6. Let $I = P(Z \geq 2) = .02275...$ where $Z \sim N(0, 1)$. Thus $I = \int h(x)f(x)dx$ there $f(x) = (2\pi)^{-1/2}\exp(-x^2/2)$ is the standard normal density and $h(x) = 1_{[x \geq 2]}$. The basic Monte Carlo estimator is $\hat{I}_1 = n^{-1}\sum_{i=1}^n h(X_i)$ where $X_1, \ldots, X_n$ are i.i.d. $N(0, 1)$. Note that most observations are "wasted" in that they are not near the right tail. Now we try again with "importance sampling": let g be the $N(3, 1)$ density. When we sample from g, the corresponding estimator of I is given by $\hat{I}_2 = n^{-1}\sum_{i=1}^n f(X_i)h(X_i)/g(X_i)$.
(a) Show that for $n = 100$

$$Var(\hat{I}_1) = p(1 - p)/n = .02275(1 - .02275)/100 = 0.0222.../100$$

where $p = I = P(Z \geq 2) = .02275....$
(b) Show that for $n = 100$

$$Var(\hat{I}_2) = n^{-1}Var_g(f(X)h(X)/g(X)) = .001805.../100,$$

and hence the variance has been reduced by a factor of 12.3.
Hint: You may compute

$$\int_{-\infty}^{\infty} \left\{h^2(x)f^2(x)/g(x)\right\} dx$$

numerically in (b).

Solution: (a) $\hat{I}_1$ is the average of the i.i.d. Bernoulli random variables $h(X_i) = 1_{[X_i \geq 2]}$ with $Eh(X_1) = P(X_1 \geq 2) = .2275 \equiv p$. Since the variance of a Bernoulli(p) random variable is $p(1-p)$ we find that

$$Var(\hat{I}_1) = n^{-2}\sum_1^n Var(1_{[X_i \geq 2]}) = n^{-1}p(1-p) = .0222.../100.$$

(b) $\hat{I}_2$ is the average of the i.i.d. random variables $w(Y_i) = f(Y_i)h(Y_i)/g(Y_i)$ where $Y_1, \ldots, Y_n$ are i.i.d. $N(3,1)$. Thus

$$Var(\hat{I}_1) = n^{-2}\sum_1^n Var(w(Y_i)) = n^{-1}Var(w(Y_1))$$

where

$$\begin{aligned} Var_g(w(Y_1)) &= Ew^2(Y_1) - \{Ew(Y_1)\}^2 = Ew^2(Y_1) - I^2 \\ &= \int_{-\infty}^{\infty} \frac{h^2(y)f^2(y)}{g(y)} dy - I^2. \end{aligned}$$

Now since

$$\frac{g(y)}{f(y)} = \frac{\exp(-(y-3)^2/2)}{\exp(-y^2/2)} = \exp(3y - 3^2/2),$$

we can write

$$
\begin{aligned}
Ew^2(Y_1) &= \int_{-\infty}^{\infty} \frac{h^2(y)f^2(y)}{g(y)} dy \\
&= \int_{2}^{\infty} \frac{f(y)}{g(y)} f(y) dy \\
&= \int_{2}^{\infty} \exp(-3y + 3^2/2) \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy \\
&= e^{3^2} \int_{2}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(y+3)^2\right) dy \\
&= e^{9} \int_{5}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz \\
&= e^{9}(1 - \Phi(5)) = 0.00232276.
\end{aligned}
$$

Thus $Var(w(Y_1)) = 0.00232276 - 0.02275^2 = 0.0018052$, which agrees with a direct numerical integration.