Math/Stat 521, Final Exam Solutions

Wellner; Friday, 12/16/2016

1. (28 points). **Define** *four* of the following six terms:

   (a) The *product* $\sigma-$*field* $\mathcal{A} \times \mathcal{A}'$ for two measurable spaces $(\Omega, \mathcal{A})$ and $(\Omega', \mathcal{A}')$.
   (b) *Almost sure convergence* of a sequence of random variables $\{X_n\}$.
   (c) *Independent* $\sigma-$*fields* **and** *independent random variables*.
   (d) The *tail* $\sigma-$*field* of a sequence of random variables $X_1, X_2, \ldots$.
   (e) *Khintchine - equivalent* sequences of random variables.
   (f) A uniformly integrable sequence of random variables $\{X_n\}$.

   Solution: See PfS, Chapters 2-8.

2. (30 points). Give careful **statements** of *three* of the following six theorems or results:
   (a) The Lebesgue decomposition theorem.
   (b) The Kolmogorov zero-one law.
   (c) The Fubini-Tonelli theorem.
   (d) Kolmogorov's Strong Law of Large Numbers.
   (e) Kolmogorov's maximal inequality.
   (f) Vitali's theorem (three parts).

   Solution: See PfS, Chapters 2-8.

3. (28 points).
   (a) State two Borel-Cantelli lemmas.
   (b) Prove the first Borel-Cantelli lemma.

   Solution: (a) Let $(\Omega, \mathcal{A}, P)$ be a probability space.
   First Borel-Cantelli Lemma: for any sequence of events or measurable sets $\{A_n\}$, $\sum_{n=1}^{\infty} P(A_n) < \infty$ implies $P(A_n \text{ i.o.}) = 0$.
   Second Borel-Cantelli Lemma: if the events $\{A_n\}$ are independent, then $\sum_{n=1}^{\infty} P(A_n) = \infty$ implies $P(A_n \text{ i.o.}) = 1$.

(b) Note that if $\sum_1^\infty P(A_n) < \infty$, then

$$\begin{aligned} P(A_n \text{ i.o.}) &= P(\cap_{n=1}^\infty \cup_{m=n}^\infty A_m) = \lim_{n\to\infty} P(\cup_{m=n}^\infty A_m) \\ &\le \lim_{n\to\infty} \sum_{m=n}^\infty P(A_m) = 0. \end{aligned}$$

4. (28 points). Suppose that $X_1, X_2, \ldots$ are independent and identically distributed random variables with $E|X_1| < \infty$.
Let $Y_n \equiv X_n 1\{|X_n| < n\}$ for $n = 1, 2, \ldots$.
(a) Show that the sequences $\{X_n\}$ and $\{Y_n\}$ are Khintchine - equivalent.
(b) What does the result in (a) imply about $P(X_n \neq Y_n \text{ i.o.})$?

Solution: (a) $\{X_n \neq Y_n\} = \{|X_n| \ge n\}$ so that

$$\sum_{n=1}^\infty P(X_n \neq Y_n) = \sum_{n=1}^\infty P(|X_n| \ge n) = \sum_{n=1}^\infty P(|X_1| \ge n) \le E|X_1| < \infty.$$

Thus the sequences $\{X_n\}$ and $\{Y_n\}$ are Khinchine equivalent.
(b) Since the $\{X_n\}$'s are independent, the result in (a) together with the Borel-Cantelli lemma imply that $P(X_n \neq Y_n \text{ i.o.}) = 0$.

5. (28 points) Suppose that $X_1, X_2, \ldots$ are i.i.d. with $E|X_1|^r < \infty$ for some $r > 0$. For $n \ge 1$ let $Y_n \equiv X_n 1_{[|X_n| < n^{1/r}]}$.
(a) Show that the sequences $\{X_n\}$ and $\{Y_n\}$ are Khinchine - equivalent.
(b) What can you say about the sequence $M_{n,r} \equiv n^{-1/r} \max_{1\le k\le n} |X_k|$?

Solution: (a) Now $\{X_n \neq Y_n\} = \{|X_n| \ge n^{1/r}\} = \{|X_n|^r \ge n\}$ and hence

$$\sum_{n=1}^\infty P(X_n \neq Y_n) = \sum_{n=1}^\infty P(|X_n|^r \ge n) = \sum_{n=1}^\infty P(|X_1|^r \ge n) \le E|X_1|^r < \infty.$$

Thus the sequences $\{X_n\}$ and $\{Y_n\}$ are Khinchine equivalent.
(b) Since $E|X_1|^r < \infty$ it follows from the SLLN that
$M_{n,r}^r = n^{-1} \max_{1\le k\le n} |X_k|^r \to_{a.s.} 0$.

6. (28 points) Let $X_1, \ldots, X_n$ be i.i.d. and suppose that $xP(|X_1| > x) \to 0$ as $x \to \infty$. For $k \in \{1, \ldots, n\}$, let $Y_{k,n} \equiv X_k 1_{[|X_k| \le n]}$.
(a) Show that $P(\cup_{k=1}^n [Y_{k,n} \neq X_k]) \to 0$ as $n \to \infty$.
(b) State Feller's Weak Law of Large Numbers.

Solution: (a) Now

$$\begin{aligned} P(\cup_{k=1}^n [Y_{k,n} \neq X_k]) &\le \sum_{k=1}^n P(Y_{k,n} \neq X_k) = \sum_{k=1}^n P(|X_k| > n) \\ &= nP(|X_1| > n) \to 0 \end{aligned}$$

as $n \to \infty$.
(b) Feller's weak law of large numbers: If $X_{n1}, \ldots, X_{nn}$ are i.i.d. for each n, then the following are equivalent:
(i) $\overline{X}_n - \mu_n \to_p 0$ for some constants μ_n.
(ii) $\tau(x) \equiv xP(|X_1| > x) \to 0$ as $x \to \infty$.
(iii) $M_n \equiv n^{-1} \max_{1 \le k \le n} |X_{n,k}| \to_p 0$.

7. (25 points). Suppose that P is the measure with density $p(x) = (1/3)1_{[0,3]}(x)$ with respect to Lebesgue measure λ (on the Borel $\sigma-$field of $\mathbb{R}$), and Q is the measure with density $q(x) = (1/3)1_{[2,5]}(x)$ with respect to Lebesgue measure λ.
(a) Is $P \ll Q$? Why or why not?
(b) Give the Lebesgue decomposition of P with respect to Q.
(c) Is $Q \ll P + Q$? Why or why not?
(d) Give the Lebesgue decomposition of Q with respect to $P + Q$.
(e) If $\phi \equiv P - Q$, find $|\phi|(\mathbb{R})$.

Solution: (a) No. Since $Q([0, 2)) = 0$, but $P([0, 2)) = 2/3 > 0$.
(b) The Lebesgue decomposition of P with respect to Q is given by

$$\begin{aligned} P &= P_{ac} + P_s \qquad \text{where} \\ P_{ac}(A) &= \int_{A \cap [2,5]} 1 \cdot dQ, \\ P_s(A) &= \int_{A \cap [0,2)} (1/3) dx. \end{aligned}$$

Note that $P_s([2, 5]) = 0$ while $P_{ac}([2, 5]^c) = 0$.
(c) Yes. If $(P + Q)(A) = 0$, then both $P(A) = 0$ and $Q(A) = 0$, so in

particular $Q \ll P + Q$.
(d) We can write

$$Q(A) = \int_A dQ = \int_A q d\lambda = \int_A \frac{q}{p+q}(p+q)d\lambda = \int_A \frac{q}{p+q} d(P+Q)$$

where $q/(p+q) = 1/2$ for on $[2,3]$, $q/(p+q) = 1$ on $(3,5]$.
(e) If $\phi = P - Q$ so that $\phi(A) = \int_Q d(P-Q) = \int_A (p-q)d\lambda$, where $p - q = (1/3)1_{[0,2]} - (1/3)1_{[3,5]}$, then $|\phi|(A) = (1/3)\int_A \{1_{[0,2)} + 1_{[3,5]}\}d\lambda$, and hence $|\phi|(\mathbb{R}) = (1/3)(2+2) = 4/3$. (Also note that

$$d_{TV}(P,Q) = (1/2)\int |p-q|d\lambda = (1/2)\{\int_0^2 (1/3)d\lambda + \int_3^5 (1/3)d\lambda = 2/3$$

which gives agreement via a homework problem.)

8. (25 points). Let P_μ denote the distribution of a $N(\mu, 1)$ random variable X on $\mathbb{R}$: thus $(dP_\mu/d\lambda)(x) = \phi(x-\mu)$ where λ is Lebesgue measure on $\mathbb{R}$ and $\phi(x) = (2\pi)^{-1/2}\exp(-x^2/2)$.
(a) Show that $P_\mu \ll P_0$.
(b) Find the Radon-Nikodym derivative dP_μ/dP_0.

Solution: (a) Suppose that $P_0(A) = 0$. But since $P_0(A) = \int_A \phi(x)d\lambda(x)$ where $\phi(x) > 0$ for all $x \in \mathbb{R}$, this implies $\lambda(A) = 0$. Then

$$P_\mu(A) = \int_A \phi(x-\mu)d\lambda(x) = 0,$$

and hence $P_\mu \ll P_0$.
(b) Now

$$\begin{aligned} P_\mu(A) &= \int_A \phi(x-\mu)d\lambda(x) = \int_A \frac{\phi(x-\mu)}{\phi(x)}\phi(x)d\lambda(x) \\ &= \int_A \exp(\mu x - \mu^2/2)dP_0(x). \end{aligned}$$

so $(dP_\mu/dP_0)(x) = \exp(\mu x - \mu^2/2)$.

9. (30 points). (a) Suppose that $H(x) = \int_{-\infty}^x h(t)dt$ where $h(t) \geq 0$ for all $t \in R$. Use the theorem of the unconscious statistician and Fubini's theorem to show that

$$EH(X) = \int_{-\infty}^{\infty} h(t)P(X \geq t)dt\,.$$

(b) Suppose that X is a random variable with values in $[1, \infty)$; i.e. $P(X \geq 1) = 1$. Let F be the distribution function of X: $F(x) = P(X \leq x)$. Use the result of (a) (or another direct application of the theorem of the unconscious statistician and Fubini's theorem) to prove the following formula:

$$E \log X = \int_1^\infty (1 - F(t)) \frac{1}{t} dt \,.$$

(c) Give an example of a distribution function F of a random variable X such that $E \log(X) < \infty$ but $E(X^r) = \infty$ for all $r > 0$.

Solution: (a) Now

$$\begin{aligned} EH(X) &= \int H(x) dF(x) = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{x} h(t) dt \right) dF(x) \\ &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} 1_{[t \leq x]} h(t) dt \right) dF(x) \\ &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} 1_{[t \leq x]} dF(x) \right) h(t) dt \\ &= \int_{-\infty}^{\infty} P(X \geq t) h(t) dt \\ &= \int_{-\infty}^{\infty} h(t)(1 - F(t)) dt \end{aligned}$$

since $P(X \geq t) = P(X > t$ a.e. with respect to Lebesgue measure λ.
(b) When $P(X \geq 1) = 1$ and $H(x) = \log x$, note that

$$H(x) = \log x = \int_1^x \frac{1}{t} dt = \int_{-\infty}^{x} 1_{[1,\infty)}(t) \frac{1}{t} dt$$

so the hypothesis of (a) holds with $h(t) = 1_{[1,\infty)}(t) t^{-1}$. Thus from (a) it follows that

$$E \log X = \int_{-\infty}^{\infty} h(t)(1 - F(t)) dt = \int_1^\infty \frac{1}{t}(1 - F(t)) dt.$$

(c) Let $1 - F(t) \equiv (\log(et))^{-\gamma}$ for $t \geq 1$ and for some $\gamma > 1$. Then for $\gamma = 2$

$$E \log X = \int_1^\infty \frac{1}{t(\log(et))^2} dt = \int_1^\infty y^{-2} dy = 1 < \infty.$$

On the other hand, for $r > 0$

$$\begin{aligned} EX^r &= \int_0^\infty rt^{r-1}(1-F(t))dt = \int_0^1 rt^{r-1}dt + r\int_1^\infty rt^{r-1}\frac{1}{(\log(et))^2}dt \\ &= 1 + r\int_1^\infty \frac{1}{t^{1-r}(\log(et))^2}dt \\ &= 1 + r\int_1^\infty \frac{e^{r(y-1)}}{y^2}dy = +\infty \end{aligned}$$

10. (30 points) Suppose that X and Y are independent random variables and that f and g are real-valued measurable functions from $(\mathbb{R}, \mathcal{B})$ to $(\mathbb{R}, \mathcal{B})$ such that $f(X)$ and $g(Y)$ are measurable. Suppose that we know that

$$E[f(X)g(Y)] = E[f(X)]E[g(Y)] \tag{1}$$

holds for $f = 1_A$ and $g = 1_B$ for sets $A, B \in \mathcal{B}$.
(i) Show that (1) holds for $f = 1_A$ with $A \in \mathcal{B}$ and a non-negative (measurable) function g.
(ii) Using the result of (i), show that (1) holds for f integrable (i.e. $E|f(X)| < \infty$) and $g \geq 0$ (and measurable).
(iii) Using the result of (ii), show that (1) holds for f integrable and g integrable (and measurable).

Solution: (i) Suppose that (1) holds for indicator functions $f = 1_A$ and $g = 1_B$. Fix a measurable set A and let $f = 1_A$. For a given non-negative, measurable function g, let $g_n = \sum_1^m a_j 1_{B_j}$ be a sequence of simple functions with $g_n \nearrow g$. Then, using the Monotone Convergence

Theorem,

$$
\begin{aligned}
E\{1_A(X)g(Y)\} &= E\{1_A(X)\lim g_n(Y)\} = \lim_n E\{1_A(X)g_n(Y)\} \\
&= \lim_n E\{1_A(X)\sum_1^m a_j 1_{B_j}(Y)\} = \lim_n \sum_1^m a_j E\{1_A(X)1_{B_j}(Y)\} \\
&= \lim_n \sum_1^m a_j E\{1_A(X)\}E\{1_{B_j}(Y)\} \quad \text{by our hypothesis} \\
&= E\{1_A(X)\}\lim_n \sum_1^m a_j E\{1_{B_j}(Y)\} \\
&= E\{1_A(X)\}\lim_n E\{\sum_1^m a_j 1_{B_j}(Y)\} = E\{1_A(X)\}E\{g(Y)\}
\end{aligned}
$$

by the Monotone Convergence Theorem again. Thus the formula holds for $f = 1_A$ and $g \geq 0$.
(ii) Now suppose that $f \geq 0$, $g \geq$ are both measurable. Then there exists a sequence of simple functions $f_n = \sum_1^p a_j 1_{A_j} \nearrow f$. Thus by the Monotone Convergence Theorem

$$
\begin{aligned}
E\{f(X)g(Y)\} &= E\{\lim_n f_n(X)g(Y)\} = \lim_n E\{f_n(X)g(Y)\} \\
&= \lim_n E\{\sum_1^p a_j 1_{A_j}(X)g(Y)\} = \lim_n \sum_1^p a_j E\{1_{A_j}(X)g(Y)\} \\
&= \lim_n \sum_1^p a_j E\{1_{A_j}(X)\}E\{g(Y)\} \qquad \text{by part (i)} \\
&= \lim_n E\{\sum_1^p a_j 1_{A_j}(X)\}E\{g(Y)\} \qquad \text{by linearity of expectation} \\
&= \lim_n E\{f_n(X)\}Eg(Y) = Ef(X)Eg(Y)
\end{aligned}
$$

by the Monotone Convergence Theorem again. Thus (1) holds for $f \geq 0$, $g \geq 0$. To extend to a general integrable function $f = f^+ - f^-$ and

$g \geq 0$, write

$$\begin{aligned} E\{f(X)g(Y)\} &= E\{(f^+(X) - f^-(X))g(Y)\} = E\{f^+(X)g(Y)\} - E\{f^-(X)g(Y)\} \\ &= Ef^+(X)Eg(Y) - Ef^-(X)Eg(Y) \\ &\qquad \text{by the result just proved for non-negative } f, g \\ &= (Ef^+(X) - Ef^-(X))Eg(Y) = Ef(X)Eg(Y). \end{aligned}$$

Thus the formula holds for a general integrable function f and $g \geq 0$.
(iii) Now let f and g be general integrable functions. Then since $g = g^+ - g^-$ we can write

$$\begin{aligned} E\{f(X)g(Y)\} &= E\{f(X)(g^+(Y) - g^-(Y))\} = E\{f(X)g^+(Y)\} - E\{f(X)g^-(Y)\} \\ &= E\{f(X)\}Eg^+(Y) - E\{f(X)\}Eg^-(Y) \\ &= E\{f(X)\}(Eg^+(Y) - Eg^-(Y)) \\ &= E\{f(X)\}E\{g(Y)\}; \end{aligned}$$

Thus the formula (1) holds for general integrable f, g.

11. (27 points). Let $X_1, X_2, \ldots$ be i.i.d. with distribution function F given by

$$F(x) = \begin{cases} \frac{1}{2}\exp(-|x|^4/3), & x \leq 0 \\ 1 - \frac{1}{2}\exp(-|x|^4/3), & x \geq 0 \end{cases}, \tag{2}$$

and let $M_n \equiv \max_{1 \leq k \leq n} X_k$.
(a) Show that

$$\limsup_{n\to\infty} \frac{X_n}{(3\log n)^{1/4}} = 1 \quad a.s.$$

(b) Show that $M_n/(3\log n)^{1/4} \to_{a.s.} 1$.

Solution: (a) Let $\epsilon > 0$. Then

$$P(X_n > (1+\epsilon)(3\log n)^{1/4}) = 2^{-1}\exp\left(-(1+\epsilon)^4 \log n\right) = 2^{-1}n^{-(1+\epsilon)^4},$$

and hence

$$\sum_{n=1}^{\infty} P(X_n > (1+\epsilon)(3\log n)^{1/4}) = \sum_{n=1}^{\infty} n^{-(1+\epsilon)^4} < \infty.$$

Since the X_n's are independent, this implies (by the first Borel-Cantelli lemma), that

$$P(X_n > (1+\epsilon)(3\log n)^{1/4} \text{ i.o.}) = 0.$$

This yields

$$\limsup_{n\to\infty} \frac{X_n}{(3\log n)^{1/4}} \le 1 \quad \text{a.s.} \tag{3}$$

On the other hand, taking $\epsilon = 0$ yields $P(X_n \ge (3\log n)^{1/4}) = 2^{-1}n^{-1}$, and hence

$$\sum_{n=1}^{\infty} P(X_n > (3\log n)^{1/4}) = \sum_{n=1}^{\infty} 2^{-1}n^{-1} = \infty,$$

so $P(X_n \ge (3\log n)^{1/4}$ i.o.$) = 1$ by the second Borel-Cantelli lemma. This yields

$$\limsup_{n\to\infty} \frac{X_n}{(3\log n)^{1/4}} \ge 1 \quad \text{a.s.} \tag{4}$$

Combining (3) and (4) we conclude that

$$\limsup_{n\to\infty} \frac{X_n}{(3\log n)^{1/4}} = 1 \quad \text{a.s.} \tag{5}$$

(b) Note that $\{M_n\}$ is non-decreasing and $M_n \ge X_n$. Thus from (a) we see that

$$\limsup_{n\to\infty} \frac{M_n}{(3\log n)^{1/4}} \ge \limsup_{n\to\infty} \frac{X_n}{(3\log n)^{1/4}} = 1 \quad \text{a.s.}$$

But from (a) we also know that for each $\epsilon > 0$ we have $X_k/(3\log k)^{1/4} \le (1+\epsilon)$ for all $k \ge$ some N_ω for all ω in a set with probability 1. Thus

$$\begin{aligned}
\frac{M_n}{(3\log n)^{1/4}} &\le \frac{\max_{k\le N_\omega} X_k}{(3\log n)^{1/4}} \vee \frac{\max_{N_\omega\le k\le n} X_k}{(3\log n)^{1/4}} \\
&\le \frac{\max_{k\le N_\omega} X_k}{(3\log n)^{1/4}} \vee \frac{\max_{k\le n}(1+\epsilon)(3\log k)^{1/4}}{(3\log n)^{1/4}} \\
&\to 0 \vee (1+\epsilon) = 1+\epsilon.
\end{aligned}$$

Since $\epsilon > 0$ was arbitrary, this yields the claim: $M_n/(3\log n)^{1/4} \to_{a.s.} 1$.

12. (27 points) Let $X_1, X_2, \ldots$ be i.i.d. exponential (1) random variables; i.e. $P(X_1 > x) = e^{-x}$ for all $x \ge 0$. Let $M_n \equiv \max_{1\le k\le n} X_k$.

(a) Show that $\limsup_{n\to\infty}(X_n/\log n) = 1$ a.s.
(b) Show that $M_n/\log n \to_{a.s.} 1$.
(c) Show that $M_n - \log n \to_d$ some Y and find the distribution function of Y.

Solution: (a) For any $\epsilon > 0$ we have

$$P(X_n > (1+\epsilon)\log n) = e^{-(1+\epsilon)\log n} = n^{-(1+\epsilon)}.$$

so $P(X_n > (1+\epsilon)\log n \ \text{ i.o.}) = 0$ by the first Borel-Cantelli lemma. Similarly, for $\epsilon in[0,1)$,

$$P(X_n > (1-\epsilon)\log n) = e^{-(1-\epsilon)\log n} = n^{-(1-\epsilon)},$$

and hence $P(X_n > (1-\epsilon)\log n \ \text{ i.o.}) = 1$ by the second Borel-Cantelli lemma. Thus we conclude that $\limsup_{n\to\infty}(X_n/\log n) = 1$ a.s.
(b) Since M_n is non-decreasing and $M_n \geq X_n$, it follows that

$$\limsup_{n\to\infty} \frac{M_n}{\log n} \geq \limsup_{n\to\infty} \frac{X_n}{\log n} = 1 \text{ a.s.}$$

But from (a), for each $\epsilon > 0$ we have $X_k/\log k \leq (1+\epsilon)$ for all $k \geq$ some N_ω on a set with probability 1. Thus

$$\begin{aligned} \frac{M_n}{\log n} &\leq \frac{\max_{k\leq N_\omega} X_k}{\log n} \vee \frac{\max_{N_\omega \leq k \leq n} X_k}{\log n} \\ &\leq \frac{\max_{k\leq N_\omega} X_k}{\log n} \vee \frac{\max_{k\leq n}(1+\epsilon)\log k}{\log n} \\ &\to 0 \vee (1+\epsilon) = 1+\epsilon. \end{aligned}$$

Since $\epsilon > 0$ was arbitrary, this yields the claim: $M_n/\log n \to_{a.s.} 1$.
(c) Now

$$\begin{aligned} P(M_n - \log n \leq x) &= P(M_n \leq x + \log n) = P(\cap_{k=1}^n [X_k \leq x + \log n]) \\ &= (1 - \exp(-(x+\log n)))^n = (1 - n^{-1}e^{-x})^n \\ &\to \exp(-\exp(-x)) \end{aligned}$$

for all $x \in \mathbb{R}$. Thus $M_n - \log n \to_d Y$ where Y has the extreme value (or Gumbel) distribution function $F_Y(y) = \exp(-\exp(-y))$.