Statistics 521, Problem Set 9

Wellner; 11/30/2016

Reading:

- Shorack, PfS, Chapter 7, pages 123 - 129;
  Shorack, PfS, Chapter 8, pages 147 - 174.

Due: Wednesday, December 7, 2016

1. PfS, Exercise 7.1.1, page 124:
   (a) Show that $P(AB) = P(A)P(B)$ if and only if $\{\emptyset, A, A^c, \Omega\}$ and $\{\emptyset, B, B^c, \Omega\}$ are independent $\sigma-$fields.
   (b) Show that $A_1, \ldots, A_n$ are independent if and only if (for each $k = 1, \ldots, n$,
   $$P(A_{i_1} \ldots A_{i_k}) = \prod_{j=1}^{k} P(A_{i_j}) \qquad \text{whenever} \ \ 1 \le i_1 < \cdots < i_k \le n\,.$$
2. Give an example of two collections of sets $\mathcal{A}_1$ and $\mathcal{A}_2$ that are independent but the generated $\sigma-$fields are not independent.
3. Show that if X_n is any sequence of random variables, there are constants $c_n \to \infty$ so that $X_n/c_n \to_{a.s.} 0$.
4. Show that if $P(A_n) \to 0$ and $\sum_{n=1}^{\infty} P(A_n \cap A_{n+1}^c) < \infty$, then $P(A_n \text{i.o.}) = 0$.
5. Let $X_1, X_2, \ldots$ be independent. Show that $\sup_{n\ge1} X_n < \infty$ almost surely if and only if $\sum_{n=1}^{\infty} P(X_n > M) < \infty$ for some $M < \infty$.
6. Suppose that $X_1, X_2, \ldots$ are independent with $P(X_n > x) = x^{-r}$ for all $x \ge 1$ and $n = 1, 2, \ldots$ with $r > 0$. Show that $\limsup_{n\to\infty} (\log X_n)/\log n = c$ almost surely for some number c, and find c.

   Optional bonus problems:

7. **Optional bonus problem:** PfS, Exercise 5.3.1, page 94.
Prove that:
(a) A function $\underline{X} = (X_1, X_2, \ldots) : \Omega \to R^\infty$ is $\mathcal{B}_\infty - \mathcal{A}-$measurable if and only if each X_n is $\mathcal{B} - \mathcal{A}-$measurable.
(b) If $\underline{X} = (X_1, X_2, \ldots)$ is $\mathcal{B}_\infty - \mathcal{A}-$ measurable and if $(i_1, i_2, \ldots)$ is an arbitrary sequence of integers, then $\underline{Y} \equiv (X_{i_1}, X_{i_2}, \ldots)$ is $\mathcal{B}_\infty - \mathcal{A}-$measurable.

8. **Optional bonus problem:** PfS, Exercise 7.1.3, page 126.
If $\mathcal{C}_1, \ldots, \mathcal{C}_n$ are independent classes of events and if each $\mathcal{C}_i$ is a $\overline{\pi}$-system, then $\sigma[\mathcal{C}_1], \ldots, \sigma[\mathcal{C}_n]$ are independent $\sigma-$fields.

9. **Optional bonus problem:** Let $X_1, X_2, \ldots$ be independent with $P(X_n = 1) = p_n$ and $P(X_n = 0) = 1 - p_n$. Show that: (i) $X_n \to_p 0$ if and only if $p_n \to 0$, and $X_n \to_{a.s.} 0$ if and only if $\sum_n p_n < \infty$.

10. **Optional bonus problem:** (a) Suppose that $X_1, X_2, \ldots$ are random variables with mean 0, $EX_j^2 = 1$, and $E(X_i X_j) = 0$ for all $i \neq j$, and let $S_n \equiv X_1 + \cdots + X_n$. Show that $S_n/n^\alpha \to_{a.s.} 0$ for any $\alpha > 1$.
(b) Suppose that $X_1, X_2, \ldots$ are random variables with mean 0, $E(X_i X_j) = 0$ for all $i \neq j$, and $\sup_j EX_j^2 < \infty$. Show that $S_n/n^\alpha \to_p 0$ for any $\alpha > 1/2$.

11. **Optional bonus problem!** Suppose $U(\omega) = \omega$ for

$$(\Omega, \mathcal{A}, P) = ((0,1], \mathcal{B}_{(0,1]}, \lambda)$$

where λ is Lebesgue measure (restricted to $(0,1]$). Thus $U \sim \text{Uniform}(0,1)$. Define

$$T(\omega) = \begin{cases} 2\omega, & 0 < \omega \le 1/2, \\ 2\omega - 1, & 1/2 < \omega \le 1, \end{cases} \qquad X_1(\omega) = \begin{cases} 0, & 0 < \omega \le 1/2, \\ 1, & 1/2 < \omega \le 1, \end{cases}$$

and, for $i \ge 2$, $X_i(\omega) = X_1(T^{i-1}\omega)$. It follows that

$$\sum_{i=1}^{n} \frac{X_i(\omega)}{2^i} < \omega \le \sum_{i=1}^{n} \frac{X_i(\omega)}{2^i} + \frac{1}{2^n}$$

and the X_i's give the diadic (non-terminating expansion) representation of U:

$$U(\omega) = \sum_{i=1}^{\infty} \frac{X_i(\omega)}{2^i}.$$

Show that $X_1, X_2, \ldots$ are independent random variables.
[Hint: see Billingsley, *Probability Theory and Measure*, pages 1-5 and A31, page 572.]

12. **Optional bonus problem!** Let $(\Omega_j, \mathcal{A}_j)$ be measurable spaces for $j = 1, 2, \ldots$. Let P_1 be a probability measure on $(\Omega_1, \mathcal{A}_1)$. Supose that for each n and each $\omega_j \in \Omega_j$ for $j = 1, \ldots, n$, $P(\omega_1, \ldots, \omega_n)(\cdot)$ is a probability measure on $(\Omega_{n+1}, \mathcal{A}_{n+1})$ with the property that for each $A \in \mathcal{A}_{n+1}$, $(\omega_1, \ldots, \omega_n) \mapsto P(\omega_1, \ldots, \omega_n)(A)$ is measurable for the product sigma-field $\mathcal{A} \times \cdots \times \mathcal{A}_n$. Show that there exists a probability measure P on the product $\sigma-$field for $\prod_{j=1}^{\infty} \Omega_j$ such that for each n and each set B in the product sigma-field $\mathcal{A}_1 \times \cdots \mathcal{A}_n$ and $C = B \times \Omega_{n+1} \times \Omega_{n+2} \times \cdots$,

$$P(C) = \int \cdots \int 1_B(\omega_1, \ldots, \omega_n) dP(\omega_1, \ldots, \omega_{n-1})(\omega_n) \cdots dP(\omega_1)(\omega_2) dP(\omega_1).$$

Hints: Prove this first for $n = 2$. Then prove it for general n by induction on n. Go from finite to infinite products as in the proof of Theorem 1 of the "Product measures" handout; i.e. Theorem 8.2.2 of Dudley (RAP), page 257. The problem is a rewording of Dudley's problem 7, page 260.