Statistics 521, Problem Set 8

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Reading: Shorack, PfS, Chapter 7, pages 123 - 129.
Shorack, PfS, Chapter 8, pages 147 - 174.

Due: Wednesday, November 30, 2016.

1. PfS Exercise 6.4.3, page 114. Prove just the parts of these formulas involving F, not the parts involving F^{-1}. You may also use Fubini's theorem directly. That is, show that:
(i) If $X \ge 0$ has d.f. F, then
$\int_0^\infty P(X > x)dx = E(X) = \int_0^\infty (1 - F(x))dx$.
(ii) If $E|X| < \infty$ then
$E(X) = -\int_{-\infty}^0 F(x)dx + \int_0^\infty (1 - F(x))dx$.
(iii) Let $r > 0$. If $X \ge 0$, then
$\int_0^\infty P(X^r > x)dx = E(X^r) = \int_0^\infty rx^{r-1}(1 - F(x))dx$.

2. Prove the two formulas in (17), PfS page 113: if $X \ge 0$ is integer valued, then $E(X) = \sum_{k=1}^\infty P(X \ge k)$ and $E(X^2) = \sum_{k=1}^\infty (2k-1)P(X \ge k)$.

3. PfS Exercise 4.9, page 114: For any distribution function F on $\mathbb{R}$ we have $\int \{F(x+\theta) - F(x)\}dx = \theta$ for each $\theta \ge 0$.

4. PfS Exercise 4.11, page 114:
(a) Show that $\int_0^\infty \{P(|X| > x)\}^{1/2}dx < \infty$ implies $E(X^2) < \infty$.
(b) Show that $\int_0^\infty \{P(|X| > x)\}^{1/2}dx \le \frac{r}{r-2}\|X\|_r$ for any $r > 2$ so that the integral on the left is finite whenever $X \in \mathcal{L}_r$ for any $r > 2$. If $\int_0^\infty \{P(|X| > x)\}^{1/2}dx < \infty$ then we say that $X \in \mathcal{L}_{2,1}$; this condition arises in connection with optimal transportation inequalities for empirical processes and in multiplier and bootstrap CLTs.

5. PfS, Exercise 5.1.4, page 91.
Let $\mathcal{X} = [0,1]$, $\mathcal{Y} = (1,\infty)$ both equipped with the Borel sets and Lebesgue measure. Let $f(x,y) = e^{-xy} - 2e^{-2xy}$. Show that
(i) $\int_0^1 (\int_1^\infty f(x,y)dy)dx = \int_0^1 x^{-1}(e^{-x} - e^{-2x})dx$ exists and is > 0.
(ii) $\int_1^\infty (\int_0^1 f(x,y)dx)dy = \int_1^\infty y^{-1}(e^{-2y} - e^{-y})dy$ exists and is < 0.

6. **Optional bonus problem:** Show that $f(x,y) = e^{-xy}\sin(x)$ is integrable with respect on Lebesgue measure on R^2 in the strip $0 < x < a$, $0 < y$. Perform the double integral in the two different orders to find that

$$\int_0^a \frac{\sin(x)}{x} dx = \frac{\pi}{2} - \cos(a)\int_0^\infty \frac{e^{-ay}}{1+y^2} dy - \sin(a)\int_0^\infty \frac{ye^{-ay}}{1+y^2} dy\,.$$

Use the inequality $1 + y^2 \geq 1$ to obtain the bound

$$\left|\int_0^a \frac{\sin(x)}{x} dx - \frac{\pi}{2}\right| \leq \frac{2}{a}\,.$$

Letting $a \to \infty$ yields $\int_0^\infty x^{-1}\sin(x)dx = \pi/2$.

7. **Optional bonus problem:** For $x \in \mathbb{R}$ and $t > 0$ let

$$f(x,t) \equiv (2\pi t)^{-1/2}\exp\left(-\frac{x^2}{2t}\right).$$

Let $g(x,t) \equiv \partial f/\partial t$. Fix $s > 0$. Show that

$$\int_{-\infty}^\infty \int_s^\infty g(x,t)dtdx = \int_{-\infty}^\infty -f(x,s)dx = -1,$$
$$\int_s^\infty \int_{-\infty}^\infty g(x,t)dxdt = \int_s^\infty \partial f/\partial x|_{-\infty}^\infty dt = 0.$$

Hint: Note that $\partial f/\partial x = -(x/t)f(x,t)$ and $\partial^2 f/\partial x^2 = (x^2t^{-2} - t^{-1})f = 2g$, and hence that f satisfies the partial differential equation $(1/2)\partial^2 f/\partial x^2 = \partial f/\partial t$.

8. **Optional bonus problem:** (A bivariate exponential distribution). Let $Y_1 \sim$ Exponential(μ_1), $Y_2 \sim$ Exponential(μ_2), and $W \sim$ Exponential(τ) be independent.
(a) Let $X_1 \equiv \min\{Y_1, W\}$, $X_2 \equiv \min\{Y_2, W\}$. Find the joint survival function $\overline{F}(x_1,x_2) \equiv P(X_1 > x_1, X_2 > x_2)$ of (X_1, X_2).
(b) Find $P(X_1 = X_2)$.
(c) Use (b) to show that the joint probability distribution $Q \equiv P_{X_1,X_2}$ of (X_1, X_2) is not absolutely continuous with respect to Lebesgue measure λ_2 on $\mathbb{R}^2$. Identify the singular component of the joint distribution Q by computing $P(X_1 = X_2 > z)$ for $z \geq 0$.
(d) What is the density of Q_{ac}, the absolutely continuous component of Q with respect to λ_2?